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THE EFFECT ON LEARNERS STRATEGIES OF VARYING COMPUTER-BASED REPRESENTATIONS: EVIDENCE FROM GAZES, UTTERANCES, ACTIONS AND SKETCHES

By
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A thesis submitted in partial fulfilment of the requirements for the award of the degree Doctor of Philosophy in Educational Technology (PhD in Educational Technology), at The Open University

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DEDICATION

Para sa aking mahal na ina at ama

sa kanilang walang patid na pagdarasal at pag-asa;

sa kanilang walang katumbas na pang-unawa at suporta;

sa pagsisilbing tunay na inspirasyon;

lubos kong inihahandog ang thesis na ito.

(Briefly translated)
This thesis is wholeheartedly dedicated
to my loving parents

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For giving me the will to keep on going

To **The ONE**

ABSTRACT

Computer-based Multiple External Representations (MERs) have been found in some cases to help and in others to hinder the learning process. This thesis examines how varying the external representations that are presented in a computer environment influences the strategies that learners choose when tackling mathematics tasks. It has been noted (Ainsworth, 2006) that learners fail to transfer insights from one representation to another. Previous work analysing video data of learners' problem-solving with computer-based MERs emphasises the need to identify which representation is being considered by a learner as utterances are made, and to examine more closely learners' movement between representations. This research focuses on the relationship between strategy and representation during learners' problem solving.

A set of analytical techniques was developed to characterise learner strategies, to identify how different computer-based MERs influence strategy choices, and to explore how these choices change over the course of task completion. Rich data were collected using a variety of technologies: learners' shifts in attention were recorded using an unobtrusive eye-tracking device and screen capture software; keyboard and mouse actions were logged automatically; utterances and gestures were video recorded; notes and sketches were recorded in real-time using a Tablet PC. This research suggests how integrated analysis of learners' gazes, actions, writing, sketches and utterances can better illuminate subtle cognitive strategies.

The study involved completion of three tasks by eighteen participants using multiple mathematical representations (numbers, graphs and algebra) presented in different computer-based 'instantiations': **Static** (non-moving, non-changing, non-Interactive); **Dynamic** (capable of animation following keyboard inputs); **Interactive** (directly manipulable using a mouse).

Having computer-based MERs available to learners provides an opportunity to use representations with which they are comfortable. A detailed analysis showed that both representation and instantiation have an impact on strategy choice. It identified differences in expression of inferences, construction of visual images, and attention to representations between different types of instantiation. One of the important findings of the research is that learners are less likely to use imagining strategies when representational instantiation is Interactive. These results may provide some explanation of how interactivity helps or hinders learners' understanding of multiple representations.

USED ACRONYMS / ABBREVIATIONS

AOI(s)	Area(s) of interest
AVI	Audio Video Interleave
CALRG	Computers and Learning Research Group
CAS	Computer Algebra System
Codec	Compressor Decompressor
DCS	Data Capture Suite
DeFT	Design, Functions, and Tasks (Ainsworth, 2006)
DVD	Digital Video Disc
ET	Eye-tracking
Fps	Frames per second
ICT	Information and Communications Technology
IET	Institute of Educational Technology
LTS	Learning and Teaching Solutions
MERs	Multiple External Representations
MPEG	Moving Pictures Expert Group
OU	Open University
PC	Personal Computer
2-D/3-D	Two-dimensional/Three-dimensional

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ADDITIONAL MATERIAL

A compact disc (CD) has been included containing example video data from one participant.

1 THE PROBLEM AND ITS BACKGROUND

1.1 INTRODUCTION

Interactive digital technology may be changing the way people learn with representations, because it allows representations to be dynamic, linked and interactive. Given this potential, an agenda is needed to address how such new representations can be matched to learning needs. For example, new technologies have been seen as a great hope for difficult-to-learn subjects such as mathematics (Fey, 1989). However, there is much to learn about how best to deploy them.

Learners have experienced difficulties with standard external mathematical representations, such as finding a pattern given a set of numbers (Zazkis and Liljedahl, 2002), graph interpretations (Dunham and Osborne, 1991) and establishing connection between a graph and a number (Knuth, 2000). Without integrating new technologies into learning with mathematical representations, it has been a challenge to completely address some of these difficulties (Dunham and Osborne, 1991).

Even when standard external representations are made computer-based, research has shown that dynamic, linked and interactive representations may carry their own additional costs (Ainsworth, 2006). For example, when standard external representations are animated, learners have been found to experience problems in focusing their attention on other kinds of external representations on the screen, thereby making them ignore those representations that are not animated (Lowe, 2003). Focus of attention to a specific standard representation on the screen due to animation could be one of the reasons why learners experience difficulty linking external representations (Jones and Scaife, 2000). Even though one of the ways to address difficulties with external representations has been to offer these representations on computer, the effects of alternative ways to present standard external representation on a computer screen have been inconsistent (Ainsworth, 1999). For instance, while there have been claims that computerising external representations help (e.g. being able to produce many mathematical graphs in a certain period of time can help learners visualise graphs, Elliot, 2000), there are also instances when computerising these external representations can hinder learning (e.g. being able to produce many graphs in a computer environment can lead to failure to interpreting all the graphs produced, Weigand and Weller, 2001). The potential changes in learning with mathematical representations that different technologies may bring need careful exploration.

This thesis is focused on strategies that people adopt in solving problems involving external representations, set in the context of learners. It is primarily about learners' strategies for using and understanding representations rather than about meta-cognitive strategies. It would be interesting to discover whether there is a pattern in the use of different kinds of strategies when people deal with complex cognitive tasks. There is a lack of research that has examined how strategies vary when instantiations involving mathematical representations are varied. This thesis examines strategies for tackling tasks with computer-based MERs (i.e. Multiple External Representations – different forms of standard external representations) presented simultaneously on a computer in a variety of instantiations, including 'Static', 'Dynamic' and 'Interactive'. The external representations are mathematical and include numbers, equations and graphs.

This thesis exploits technology not only as a means of giving learners alternative representations, but also as a means of capturing the learners' interactions with those representations. Recent developments in digital data capture technologies have opened up new possibilities for researching the potential pedagogical benefits of computer-based representations. This research applies digital data capture to investigate systematically the effect of different computer-based representations on learning. The approaches taken to analysing strategy in this research have been inspired by the methodological challenges that the author faced in a previous study (San Diego, 2003) associated with 'traditional' data capture technologies. The methodological innovation has included a set of techniques for analysing computer interactions from eye-tracking, tablet PC screen capture, digital voice and image record, and capture of computer actions as 'logs'. The digital approach to analysing learner-computer interactions is illustrated in detail. The kinds of rich data captured and the forms of data 'triangulation' are discussed.

This chapter begins with a definition of key terms in order to set the context of the research. It is important to make a distinction between standard external representations, computer-based representations, and instantiations because these key terms appear several times in this thesis. The need for examining strategies with computer-based representations, interactivity and instantiations are then presented. The problem statement, research question and hypotheses are identified form the previous literature about learning with representations, instantiations, strategies and cognitive load. A brief illustration and discussion, of a 2003 study conducted by the author that motivates this thesis to examine strategies in detail and to explore the opportunities offered by digital technologies in order to observe strategies more accurately than traditional observation or analogue technologies (San Diego, 2003). The chapter ends with an overview of the structure of the thesis.

1.2 THE CONTEXT OF STRATEGY, REPRESENTATIONS AND INSTANTIATIONS: DEFINITION OF TERMS

The key terms used in this research are defined in detail. Each term is introduced by an example, in order to help put the terms into context.

Standard external representations

When more than one external representation is presented to learners, they are engaging with MERs. To illustrate, the external representation '–5' (as a number) corresponds to another external representation e.g. a graphical representation, a point on a line five units to the left of zero (as in Figure 1-1).

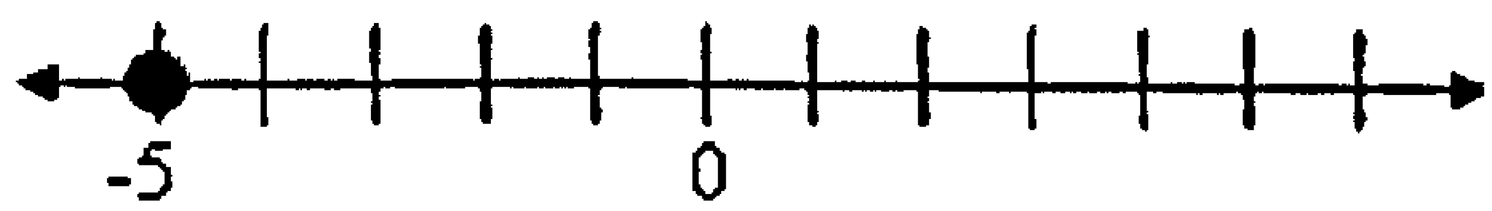


Figure 1-1 An example of a mathematical graph representation

One particular concern of interest to this research is the way people deal with representations. Representation can be defined as "something that stands for something else" (Palmer, 1977, p. 262); "a structurally equivalent presentation and an alternative presentation or something in place of something else" (Pape and Tchoshanov, 2001, p. 120). This definition can be expanded:

> *"a configuration of some kind that, as a whole or part by part corresponds to, is referentially associated with, stands for, symbolises, interacts in a special manner with, or otherwise represents something else." (Goldin and Kaput, 1996, p.398)*

The representations chosen in this research are numbers, graphs and equations – mathematical representations which have been found to be difficult to learn with as discussed later in section 1.3. These three types of representations are referred to in this research as 'standard external mathematical representations'.

Instantiations

When standard external representations are presented in a computer environment, representations may be presented in different 'instantiations'. For example, the graphical point '– 5' can be defined as an absolute distance of a point on a number line five units to the left of '0'. This can be presented as a 'moving' graphical point from '– 1', to '– 2', to '– 3', to '– 4', to '– 5'. This kind of presentation is referred to by others as an external representation that is 'animated' or as 'dynamic representation' or as moving images. This can also be presented to learners as 'freeze frames' or non-moving images based on paper (others refer to as 'static representations') (see Figure 1-2). The question is: is the 'moving graphical point' a variant of images of static graphical points, or is it a different representation altogether? However, '– 5' as a number (i.e. an external representation) can be represented as a graphical point on a number line (i.e. as another external representation).

Rogers (1999) provides an example that illustrates why the concept of 'instantiation' is valuable. In a commentary on interactive graphical representations, Rogers stated that static diagrams are effective in problem solving, but that interactive versions of the same diagrams are even better in problem solving. She went on to compare static and interactive variants using the terms 'static graphical representation' and 'interactive graphical representation'. Different instantiations may have different effects on learning, just as different representations may have different effects on learning.

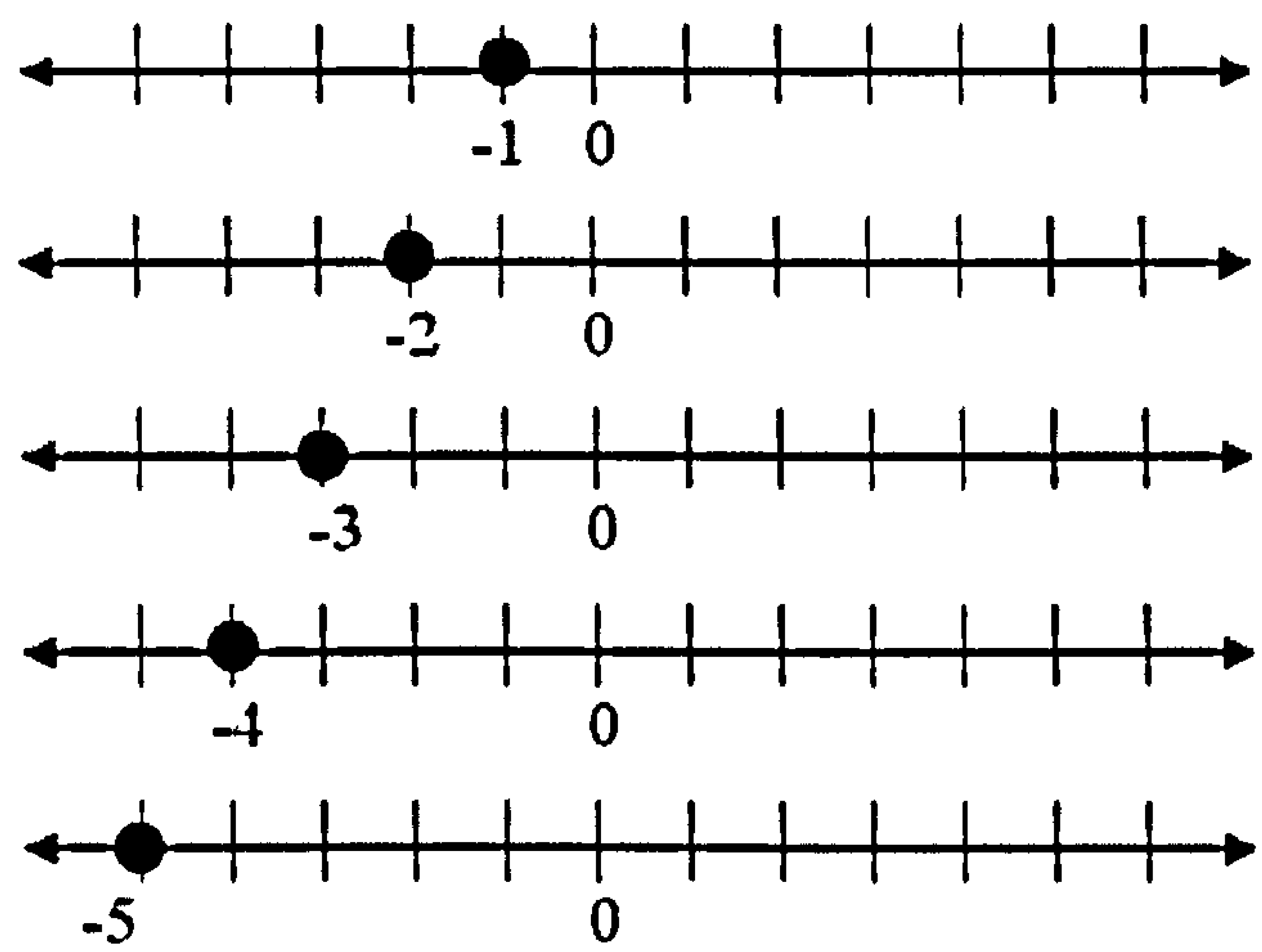


Figure 1-2 An example of capturing an animated instantiation using static instantiations

Goldin and Kaput (1996) label alternative ways to display representations as 'instantiations'. There seems to be a need to characterise instantiation and representation. For example, Goldin and Kaput propose three basic distinctions of the fundamental features of media 1) Dynamic versus Static media – one presenting motion whilst the other is still; 2) Interactive versus inert media – the former changing the display and meaning of the display through input whilst the latter only changes in display; 3) Recording versus nonrecording media – one capable of recording whereby it can be retrieved whilst the other cannot. Lim, Bensabat and Todd (1996) categorise instantiations in a different way. They distinguish 'direct' and 'indirect' manipulation: 1) Direct manipulation allows learners to, for example, drag icons within the interface; whilst 2) Indirect manipulations require them to click a series of icons.

An alternative way to categorise instantiations has to do with operational exchange. Operational exchange refers to functional activity: the entering of information through a keyboard and the resultant response from the system, (Luckin and Du Boulay, 2002, p.99) which others refer to as 'interactivity', i.e. a situation when "a user who has access to a range of input devices which can activate the technology being used; the result of this action is some form of visual output: text, graphics, printing; and the sequence of actions form an interaction." (Sims, 1997, p. 159). In Figure 1-1, for example, imagine that by using a computer's mouse, the graphical point can be dragged along the line.

Degree of interactivity – of operational exchange – is one of the defining parameters of the three instantiations used in this research. The varying instantiation has no agreed typology as yet. Nonetheless, there are proposals that seem to include as elements of instantiations: motion of representations on screen (Goldin and Kaput, 1996); manipulability of elements of representations (Lim, Bensabat and Todd, 1996); different ways of manipulating representations on the screen depending on devices (e.g. Sims, 1997); different output of representations on the screen based on operations with different input devices (e.g. Luckin and Du Boulay, 2002).

For the purposes of this study, three types of instantiation are used:

i. **Static**: Non-moving, non-changing, non-interactive;

ii. **Dynamic**: Capable of animation through alpha-numeric inputs;

iii. **Interactive**: Directly manipulable graphs.

It is worth emphasising that other types of instantiations are possible. Also, there are other ways of implementing the instantiations used in this research. This thesis will hope to demonstrate that the distinction between instantiation and representation facilitates the explication and attribution of effects and is likely to improve the clarity of resultant theory.

Strategy

Given a particular problem, involving standard external mathematical representations, people use a set of strategies in solving them. Some research attention has been given to generic strategies that learners use in solving mathematical problems (e.g. guess-and-test, working backwards, or trial-and-error; see e.g. Boulton-Lewis, 1998; Tabachneck, Koedinger and Nathan, 1994) and to strategies that relate to specific external mathematical representations (e.g. adding, counting, or sketching). When a mathematical problem is set in the context of interaction with computers, additional strategies may apply. See, for example, the strategies afforded by rapid graph plotting, described by Weigand and Weller (2001).

Aczel (2006) used the term 'strategic theories' (introduced in Aczel, 1998) as:

> *"Students' conjectured constructions of some sort of reality under the selection pressures provided by 'concerns' (problems of special interest to the student)." (p. 159)*

Aczel expands on strategic theories that strategies chosen to address a particular concern may require several attempts. Thus, according to Aczel, people subject strategies to 'trial-and improvement'. So, for example, by attempting to address a particular problem learners may produce some theories and by trying each of the theories a more improved theory than a previous tried one is produced; learners may discontinue testing out other theories when the particular problem is deemed addressed. This research focuses on strategy as attempts people make in solving a problem, in line with the strategic theories of Aczel (2006).

To illustrate how strategic theories apply to this research, take for example the representation in Figure 1-1. When learners solve a problem involving a mathematical representation, learners may choose a strategy to deal with the particular form of external mathematical representation (e.g., in this case, a line graph). Should this attempt fail, learners may attempt to improve this strategy by combining it with another form of external representation (e.g. numbers). The strategies that learners use may or may not differ between the two types of external representations involved. The study in this research involves participants engaging with a problem involving mathematical representations, and their strategies are studied in relation to those representations.

1.3 EXAMINING STRATEGIES WITH COMPUTER-BASED MERS

This section explains why it is important to examine strategies with computer-based representations. It draws attention to the significance of choosing standard external mathematical representations, among others. It outlines how the problems with standard external representations can be distinguished from the problems related to instantiations. It presents the rationale to understand how strategies evolve when learners deal with computer-based and multiple external mathematical representations as to identify success and failure in solving problems.

Multiple external representations presented in a computer environment are found in some cases to help and in others to hinder the learning process (Ainsworth, 1999). MERs include different 'standard external representations' presented together (e.g. text and picture or numbers and graphs) and the types of presentation (e.g. static and dynamic). However, as will be seen, there are unique difficulties associated with each of the standard external representations and with the types of instantiations. These representations, when presented in a computer environment, require manipulation and interaction; and of course, different types of presentation can require different actions. It is a challenge to be able to examine and analyse what learners do with external representations and how they make sense of them. Learning with different types of external representation and the different forms of presentation of these external representations is complex and is not yet fully understood. One of the possible ways to look at learners making sense of representations is by analysing strategies. Learners' choices of strategies are still somewhat neglected.

Standard external mathematical representations

Mathematical representations are used in many domains like Physics, Engineering, Chemistry, Statistics, and so on. There are some specific cases where persistent difficulties with mathematical representations have been identified. For example, learners are known to experience some difficulty with: visualising transformation of graphs (Borba and Confrey, 1996); imagining some mathematical concepts in their own mind (Sierpinska, 1994); graphing, e.g. extracting the x and y coordinates of a graph (Dunham and Osborne, 1991); linking multiple mathematical representations (Elliott, Hudson and O' Reilly, 2000; Even, 1998; Keller and Hirsch, 1998; Ruthven, 1990; San Diego, 2003; Sierpinska, 1994; Villarreal, 2000; Weigand, 1991; Hennessy, Fung and Scanlon, 2001; Knuth, 2000). Consequently, learning with three representations – numeric, algebraic and graphic – is worth studying.

Technology and mathematics

There are strong beliefs that technology is having a great impact on the learning from computer-based representations. The presence of technology may be able to make learning with mathematical representations meaningful. For example, some have stated that one of the main strengths of technology is that it can provide greater and easier access to multiple representations of concepts and processes (Goldenberg, 1988; Kaput and Thompson, 1994; Fey, 1989). Offering external representations in a computer environment has been explored with the aim of reducing some of the problems learners experience with external representations (Ainsworth, 1999). For example, in a computer environment, graphing technology can be an effective tool for enhancing learners' graphing abilities (Scanlon, 1998) such as its capability of giving immediate feedback and providing rapid graph-plotting.

Instantiating representations

There are different kinds of instantiations (see section 1.2) and technology has numerous capabilities for instantiating representations. Figure 1-3 shows some examples of instantiations.

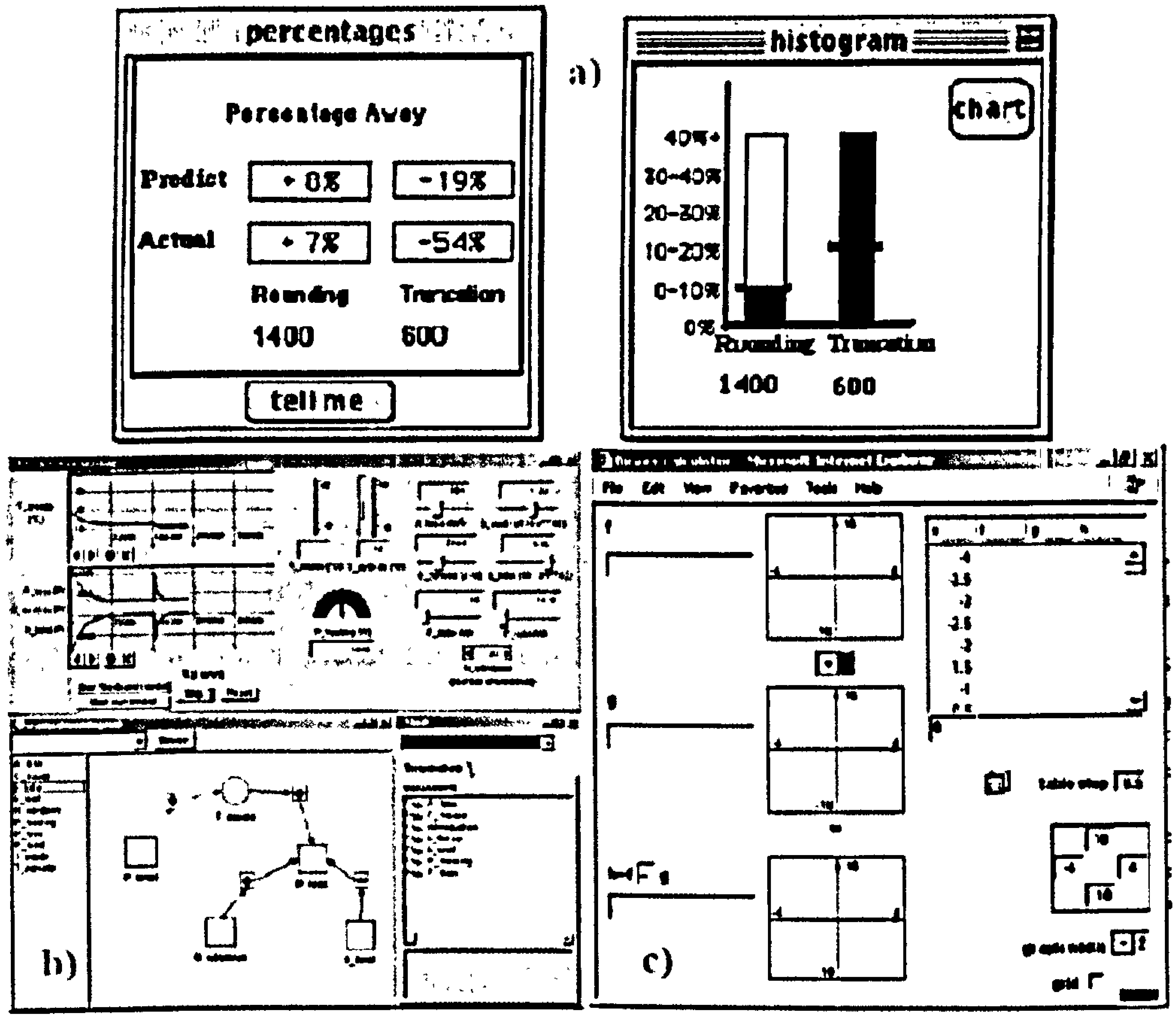


Figure 1-3 Examples of computer-based representations

The varying ways of instantiating external representations can include:

- Static (not moving around the computer screen)
- Dynamic (representations move around the screen)
- Interactive (representations that are 'manipulable' e.g. using a mouse to make them move around the screen)
- a combination of each of Static, Dynamic and Interactive
- 3-D vision (e.g. using *binocular stereoscopic techniques*),

- being able to make representations move using a stylus, insofar as this is different from using a mouse.

While there are difficulties related to standard external representations, the way these representations are instantiated adds other kinds of difficulties (e.g. Ardac and Akaygun, 2005, Lowe, 2003). For example, learners may have problems focusing their attention with external representations moving around the screen (Jones and Scaife, 2000). The speed of animation has also been found to have different effects (Van der Meij and de Jong, 2006; Ardac and Akaygun, 2005; Morrison and Tversky, 2001). These are just a few of the reasons why it is important to examine different types of instantiations.

Cognitive processing of computer-based representations

Larkin and Simon (1987) wrote

> *"One may be able to use the information in a verbal description to draw or image a diagram or use a diagram to infer verbal statements" (p.66).*

There is a difference for learners between deriving information from one representation and deriving information from multiple representations according to Scaife and Rogers (1996). Also, there are concepts which can be learned or perhaps understood best using only one particular instantiation or combinations of instantiations (Ainsworth, 2006). Different types of external representations and different types of instantiations have been claimed to influence cognitive processing, because they impose different cognitive demands (e.g. Lewalter, 2003; Lowe, 2003; Rogers, 1999; Scaife and Rogers, 1996).

There are theories to explain the optimal choice of representations and instantiations in designing learning with computers. One of the theories is related to 'cognitive load' (Sweller and Chandler, 1991). For example, reducing extraneous cognitive load (i.e. mental effort imposed by the way information is presented externally (Bodemer and Faust, 2006) such as decreasing unnecessary visual search processes (Sweller, Van Merrienboer and Paas, 1998) does not necessarily improve learning. Split-attention effects, where a particular representation, for example, can be animated whilst other representations are static, may influence learners' attention (Mayer and Moreno, 1998). Previous research has explored the role of cognitive load in explaining the effects of giving learners' alternative representations and instantiations. But it has not yet been fully explained how learners' choice between alternative representations is affected by different instantiations of those representations.

Reducing problems with computer-based representations

Although there are problems particular to standard external representations and also problems particular to specific instantiations, learning in many instances is being designed, by teachers or educational software designers, with some assumptions of how difficulties with external representations can be reduced. For example, there are studies that claim that varying external forms of mathematical representations have different effects on learning mathematics concepts, either by offering graphs or numbers (Weigand, 1991). Also, offering only one kind of external representation (Kalyuga, Chandler and Sweller, 1999) or a number of external representations (Ainsworth, Bibby and Wood, 1997) can both be found to have positive effects. However, designs can also include decisions about varying instantiations. There are inconsistencies with the impact of instantiations on learning. Elliot et al (2000) claim that an instantiation that provides rapid graph-plotting can help in reducing difficulties in visualising graphs; while Weigand (1991) claim that learners have difficulty interpreting many graphs produced by rapid graph-plotting. There is a need to identify what is different about the various technologies and what those differences measure in terms of cognition and learning. However, the process of varied interaction with these representations is not yet fully understood. Therefore, what learners make sense of – and how they make sense of – each particular representation when instantiated differently is important to examine.

It is not just the design of learning that needs to be considered in examining learners' interaction with computer-based representations. Learners themselves sometimes create their own forms of representations, different from what is given on a computer screen

(Borba and Confrey, 1996). For instance, when the design of computer-based representations does not always meet learners' own preference for representation (Weigand and Weller, 2001), they may dismiss working on the computer-based representations. This may lead to working in their head with the kinds of representations with which they are comfortable or they may use a different technology, e.g. pen and paper, to construct other external representations (Cox, 1996). Learners have their own ways of coping with the problems they experience with external representations. According to Cox (1996), when learners construct their own representations, this could signal either difficulty or insufficiency of information on the representations given. Further investigation is needed into the relationships between (i) learners' self-constructed representations; (ii) learners' ways of using computer-based representations; and (iii) the ways in which given representations are instantiated.

Examining strategies

Most of the previous research about computer-based representations has been pair-wise comparisons of representation use, or pair-wise comparison of instantiations, there is no known study that investigates combined presentation in a way that picks apart the influence of each of the representations and instantiations. One way of examining learners' interaction with computer-based representations is by looking at learners' strategies (as defined above). Learners' choice of strategies with different computer-based representations is under-researched. There is no known empirical study that investigates how, when learners are given multiple representations, their choices of strategies are influenced by type of instantiation.

Learners are using a wide range of strategies for different standard external mathematical representations (e.g. Keller and Hirsch, 1998; Ruthven, 1990; Even, 1998; Villarreal, 2000; Tabachneck et al., 1994; Novick, 1990; etc.). The strategies learners choose may depend on the kind of external representations given or they may even depend on the kinds of representations they construct on their own. Learners have also been found to use strategies for constructing external representations not on the computer screen. They have been found to externalise visual forms of representations using pen and paper (Cox, 1996), gestures (Goldin-Meadow and Wagner, 2005), and gazes (Yoon and Narayanan, 2004). The investigation of these kinds of strategies may elucidate why and how difficulties with learning from computer-based representations occur. Examining strategies may provide further explanations about learning with computer-based representations and the complex relationships of representations, interactivity, instantiations, and cognitive load. However, few attempts have been made to study learners' strategies with computer-based representations, even though clearly there is a need for investigating them.

Some studies have examined learners' strategies with one form of mathematical representation. Aczel (1998) analysed algebraic concepts using strategies. Though Aczel used this technique to investigate algebraic abilities and did not include graphical representations, the task was in a computer environment. Aczel examined learners' thought processes with algebraic representations and identified how strategies improve during the course of a task. The task in Aczel's study required certain manipulations of instantiation – that is inputting numbers which then appeared in the computer screen as 'barrels' on a weighing scale. Learners then improved their strategies in dealing with 'equations' from the weighing scale representation. The kinds of improvement in learners' strategies through a particular instantiation of scaffolding linking the numeric representations to algebraic representations were found to increase learners' understanding of equations.

This thesis investigates learners' strategies for using multiple computer-based representations. In particular, the focus is on how the way standard mathematical representations are instantiated can affect the strategies that learners' choose.

1.4 THE PROBLEM STATEMENT, RESEARCH AIMS AND HYPOTHESES

The problem statement

Learners' choice of strategies with different computer-based representations merits further research. There is an absence of empirical studies that investigate how different external representations presented in a computer environment influence learners' choice of strategies when tackling tasks using these representations. In chapter 2, it will be established that one explanation of the uneven results in applying computer-based MERs is that strategy, representation and instantiation interact. Further work is required to explicate this interaction; in order to understand precisely why difficulties arise and precisely what influences learners' choice of strategies.

There are external representations that share the same information (and therefore are not believed to be mutually exclusive) but present it in a way that encourages different inferences (and therefore are not equivalent substitutes for each other) (Larkin and Simon, 1987). In mathematics, it is often the case that such representations are presented together in order to support problem solving because they are understood to complement each other (Ainsworth, 1999). So, for example, equations and graphs are presented together, perhaps with tables of values. Thus, these external representations cannot be studied in isolation, because we need to understand the overall relationship between representation and inference. This poses some methodological challenges. In particular, it requires that the researcher be able to associate actions and inferences precisely with specific representations. Although there have been pair-wise comparisons of representation use, there is no known study that investigates combined presentation in a way that picks apart the influence of each of the representations. It is important to know which exact representation learners are considering when they choose a certain strategy. In this thesis, an innovative set of analytical techniques is developed that details strategies that learners choose in tackling tasks with computer-based MERs. Possible explanations are identified as to how different presentations of MERs influence learners' choice of strategies. The thesis explores further how these choices change over the course of a task.

Aims of the thesis

The main aim of this thesis is to examine the relationship between learners' strategies and various computer-based representations. The research investigates how different external representations presented in a computer environment influence learners' choice of strategies when tackling tasks using these representations. It also examines how different instantiations influence learners' choice of strategies. It examines how participants' strategies change over the course of a given task. By examining learners' strategies with computer-based MERs, it is hoped that this research will inform the design and effective use of computer-based MERs for learning graphing and linking multiple representations. A contributing aim is to develop a set of techniques that are effective in researching learning with computer-based MERs and so might provide some explanations as to why learners experience difficulty with computer-based MERs.

The main research question and hypotheses

The main research question is: **How do representations instantiated in different ways influence learners' cognitive processes?** To answer this, a variety of hypotheses are formulated in this thesis.

In chapter 2, five hypotheses are generated using the research literature about strategies, and computer-based representations while in this section, a hypothesis is generated about the value of looking at gazes, utterances and sketches:

1) Strategies with each standard external representation can be characterised at different levels of granularity.

2) Learners' choice of strategies depends not just on the standard external representations given but also on the instantiation.

3) Mental constructions of images with graphical representations vary between instantiations.

4) Attention paid with each standard external representation varies between instantiations.

5) Expression of inferences varies depending on the instantiation

6) Analyses of strategies based on gazes, actions, utterances and sketches can identify factors contributing to strategy choice in a way that is not possible with traditional observation techniques.

1.5 THE MOTIVATION FOR THE STUDY

In 2003, the author conducted a two-stage study on how students make conjectures based on external mathematical representations (numerical tables, graphs, and equation) generated by two different technologies (calculator and computer) (San Diego, 2003). The purpose of the study was to examine the role of multiple electronic representations in solving problems. Mathematics problems were chosen firstly because representation has traditionally played an important role in mathematical reasoning, supporting both inference (reasoning from evidence) and conjecture (forming a conclusion from incomplete evidence, making an informed guess). Secondly, mathematics problems were chosen because the representations in mathematics are in common use in teaching, in everyday life, and in other disciplines such as science and engineering. The study was seen as a first step; further research could look at the same representations in other contexts, such as science or engineering problems. The study hinted that there were changes in the participants' inferences as they moved from one technology to another (i.e. graphical calculator and graphical software).

The first stage of that study involved sixteen A-level students using a graphical calculator with pen and paper to tackle a number of problems. This was conducted in a classroom. For each set of problems, they were asked to make inferences leading to a generalised conjecture. Then, in the second stage, conducted in a laboratory, four students were identified as typical of the sixteen students in terms of the representations they used. These four students worked on a computer with graphical software as well as pen and paper. Again, for each set of problems, they were asked to make inferences leading to a generalised conjecture. In the laboratory setting, four video streams ('quad-image' in

Figure 1-4) were recorded simultaneously and combined into a single stream: two video streams were of the students' work on 'pen and paper', another video stream captured their actions with the mouse and keyboard, and the fourth stream was of the computer display.

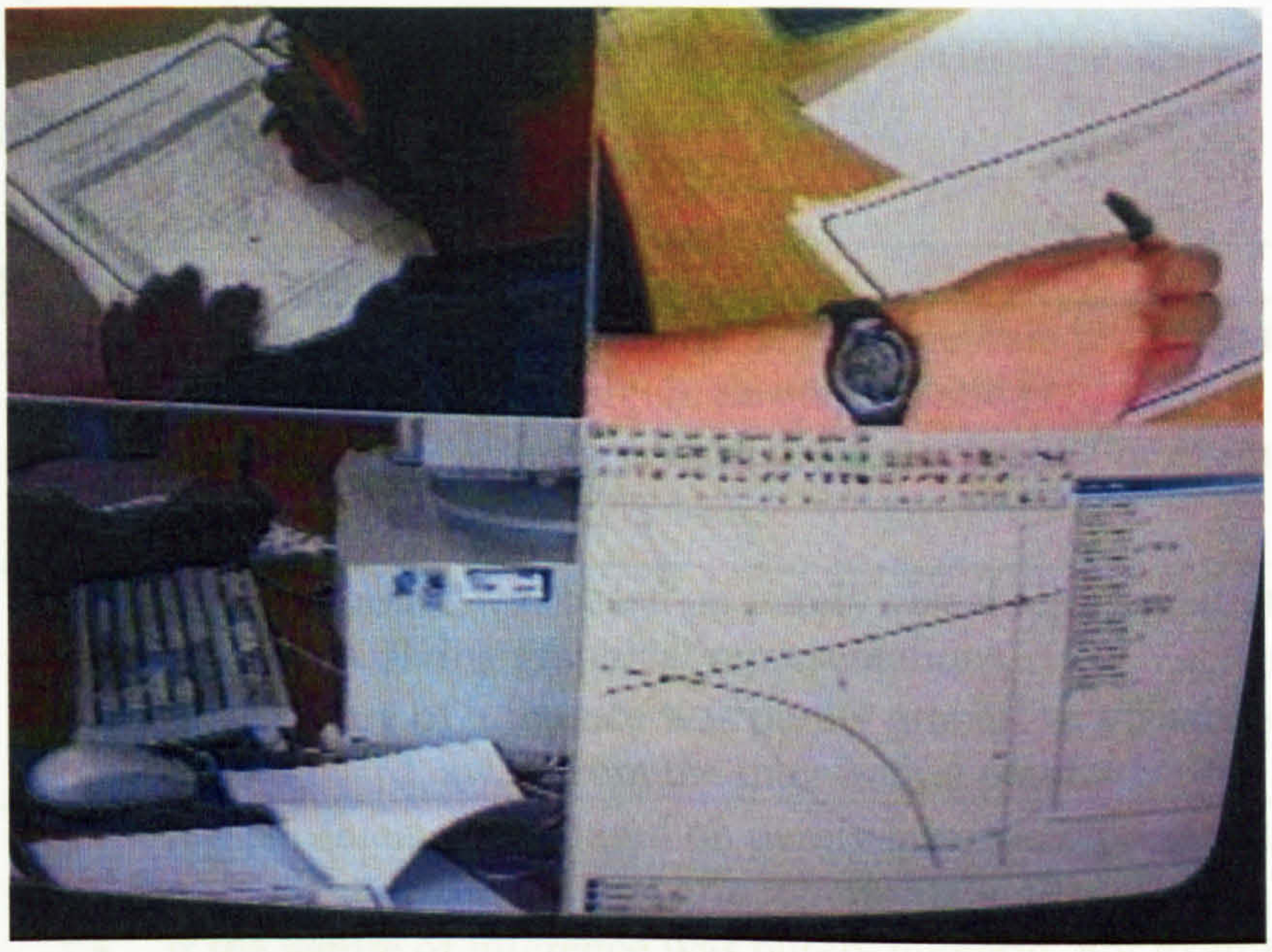


Figure 1-4 The 'quad-image' using analogue technologies

The data were analysed using a coding scheme that characterised the type of inferences the students were using as they made conjectures. These were identified by triangulating what students were seeing on the screen as they made their conjectures, and their 'think-aloud' descriptions of their conjectures. Table 1-1 shows some examples of the inference codes.

A weakness of the study was that it was difficult to tell exactly which representations students were using as they made conjectures. Another weakness was that while the data included the sketches or notes that students made when tackling the problems, they could be analysed only as post hoc traces rather than as real-time aspects of the students' use of representations. It was found difficult to integrate systematically the on-screen representations, the utterances of the students and their actions. The coding scheme was also found to be limited in understanding the rich range of students' behaviours. Nevertheless, the study gave rise to a hypothesis that differences between kinds of conjectures made by given students may have been attributable to the way these external representations were presented by the two technologies. The differences in conjectures may be due to the different manipulation required and the different external representations offered by the two technologies, rather than the technology themselves. The results need further investigation, in order to test this hypothesis.

Table 1-1 Examples of inference codes

Inference code	Characterisation of inferences
Purely Algebraic	• If the inference was based on purely algebraic representation or from the activities involving algebraic manipulations. • If the nature of inference was purely algebraic. For example: $If\ x=0,\ then\ x_n \to 4\ as\ x \to \infty$
Purely Visual	• If the inference was based on purely visual representation or from the activities involving graphical representations. • If the nature of inference was purely graphic or visual. For example: It goes upward, away from the intersection point.
Purely Numeric	• If the inference was based on purely numeric representation or from the activities involving numeric representations • If the nature of inference was purely numeric. For example: The sequence repeats, 0, -1, 0, -1...

The procedures and coding scheme used in the 2003 study have been adopted and extended in the research presented in this thesis (see Chapter 3). The conclusions of that study also led to the focus in this thesis on the influences of how representations are presented and on the strategies that learners choose when using these representations.

1.6 USING DIGITAL APPROACHES TO EXAMINING STRATEGIES

Some of the questions that arose from the 2003 study described in section 1.5 above are taken up in this doctoral research. In particular, the methodological limitations of the 2003 study motivated deployment of a suite of new approaches to data analysis and collection, in order to examine the relationship between strategy and representation more precisely, and in order to take more kinds of evidence (e.g. notes, gestures) into account in the analysis. State-of-the-art technologies were used to improve data capture and analysis. The following variety of research techniques was chosen and combined in order to enable an investigation of what learners are seeing, in relationship to what they say, do and write.

It is important to know which exact representation learners are considering when they choose a certain strategy. One of the techniques to use is 'think-aloud' protocol analysis (Ericsson and Simon, 1984). This is a procedure to ask people to talk aloud while going through a task given. In computer-interaction studies, many researchers have recorded screen activity and voice (e.g. Aczel, 1998; Weigand and Weller, 2003; San Diego, 2003; Scanlon et al., 2005) but often utterances include ambiguous signifiers such as 'this', 'here', 'that', etc. In examining use of multiple external representations, these ambiguous signifiers have to be identified and matched with the exact representation. Techniques that can help reduce this problem include the use of eye-tracking protocol (e.g. Hansen, 1991; Eger, 2005) and real-time video capture of sketches (e.g. Pirie, 1996) as there are representations that can be difficult to verbalise (Cox, 1996). Also, by capturing sketches, it can be important to be able to tell exactly at which point in time a person 'erases' something and the 'sketch' that was erased. This is not possible by post-hoc analysis of sketches paper alone.

Although these techniques may help reduce some of the methodological challenges in using traditional video capture and analysis, the capture of simultaneous videos of screen activity, a person's actions, gazes and sketches also poses new methodological challenges such as coordination of videos and analysis of rich accounts of these videos. In this thesis, an innovative set of analytical techniques is developed. It provides a detailed illustration and discussion about the affordances of using digital approaches to capturing and analysing learners' computer interactions in detailing strategies that learners choose in tackling tasks with computer-based MERs.

1.7 OVERVIEW OF THE THESIS

An overview of each chapter is presented in this section.

Chapter 1: The problem and its background

Chapter 1 has outlined the rationale for researching learners' strategies with computer-based representations, to be discussed further in Chapter 2. It has defined key terms used in this thesis and has briefly described the 2003 study that motivated the development of new data capture and analysis techniques with digital records of gazes, actions, utterances and sketches. These techniques are described in detail in Chapter 3.

Chapter 2: Representations, instantiations and strategies

Chapter 2 takes account of associated background theories related to external cognition and cognitive load. It presents evidence from previous empirical studies about the influences of different types of instantiations on representations. It discusses the conflicting claims, difficulties and strategies associated with multiple representations, instantiations, visualisations and constructions of inferences. The review critically assesses key research relevant to the area being investigated in this thesis. It identifies the gap in the literature relating the influences of varying instantiations on representations and explores how this gap may be examined.

Chapter 3: Innovative approaches to capturing video data of learners' computer interactions

Chapter 3 considers how digital approaches to capturing video data of learners' computer interactions can minimise some of the methodological problems of recent technologies. It illustrates the combination of eye-tracking, Tablet PC screen capture and digital video in capturing learners' gazes, actions, writing, sketches and utterances. It also outlines the reasons for choosing video to research learners' computer interactions, and the typical challenges faced.

Chapter 4: Capturing learners' utterances and actions, with multiple representations

Chapter 4 focuses on the data collection techniques that have been selected to help investigate the research hypotheses. It gives an explanation of the research design undertaken for the study. It presents the design of tasks, representations, materials, time and setting, and procedures involved in the study.

Chapter 5: Approaches to analysing learners strategies with multiple representations

Chapter 5 describes the software analysis tools used in the study. It describes the phases of analysis and illustrates some examples of analysing gazes, utterances, actions, and sketches. It presents the categorisation and characterisations of strategies with different mathematical representations, expression of inferences and imagery. It illustrates how the coding schemes developed are operationalised and shows an example of the kinds of analyses with gazes, actions, utterances and sketches. It also discusses the validation of the coding schemes.

Chapter 6: Quantitative data: a preliminary analysis

Chapter 6 presents the analysis of quantitative data such as time to completion of tasks, areas of interest analysis, performance of participants in each task, and frequency of chosen strategies. It discusses the findings in the light of the connection of time to task completion to choice of strategies and performance of the task.

Chapter 7: Integrated analysis of gazes, utterances, sketches and actions

Chapter 7 presents and discusses the integrated analysis of gazes, utterances, actions and sketches. It relates the results of the analyses to the research hypotheses and provides a number of alternative explanations for these results. This chapter also illustrates the analysis of changes in strategies with multiple representations. It provides some examples of participants' accounts of strategies. It also presents some of the other incidental findings related to difficulty with mathematical representations and interactivity.

Chapter 8: General discussion, implications and future research

Finally, chapter 8 relates the findings of the study to previous literature about representations, instantiations and strategies. It also discusses the limitations and implications of the study. It gives some recommendations for future research.

2 REPRESENTATIONS, INSTANTIATIONS, INTERACTIVITY AND STRATEGIES

2.1 INTRODUCTION

The power of the computer to allow learners to automatically produce different external forms of representations, to make sense of given representations, and to be able to manipulate them has been perceived by some as the 'holy grail' for making mathematics learning effective. However, the evidence of the benefits of computer-based representations is mixed. This chapter reviews previous research with multiple computer-based representations.

The key terms as defined and detailed in section 1.2 are referred to several times in this chapter. It is important to establish the differences between e.g. standard external representation, instantiation, computer-based representations, multiple external representations, and multiple computer-based multiple representations. Computer-based representations are characterised by the information presented, by the form in which this externalised information is presented, and by the nature and degree of interactivity of that presentation. This thesis refers to the form of presentation as the 'standard external representation' and to the variations in the types and degree of interactivity and varying 'instantiations'. This chapter makes a case that there are issues associated both with external representations and with the type of instantiation. The kinds of representations chosen to illustrate this case are set in the context of learning with mathematical representations. By investigating previous research about learners' experiences with mathematical representations and the attempts they use in solving problems with these representations, this thesis hopes to argue for the significance of looking at the effect on strategies of varying computer-based representations.

Table 2-1 Framework of the literature review

Difficulties	**Theories**	**Effects due to combination of external representations**	**Reducing difficulties**	**Characterisations of strategies**
Performing operations with algebraic symbolisations and finding a patter in a set of numbers (Zazkis and Liljedahl, 2002) Establishing connections between graphs and numbers (Knuth, 2000) Inattention to scales, extracting numerical information from graphs, views of transformations of graphs (Dunham and Osborne, 1990) Mentally elaboration from static display (Ardac and Akaygun, 2005) Extracting information on complex animations (Lowe, 2003) Focusing attention in animation (Jones and Scaife, 2000) Focusing on manipulable elements (Otero, Rogers and du Boulay, 2001)	Computational off-loading, re-representations and graphical constraining (Scaife and Rogers, 1996) Intrinsic and extraneous cognitive load (Sweller and Chandler, 1991) Split-attention effect (Mayer and Moreno, 1998) Redundancy effect (Kalyuga et al., 1999)	External representations offered influences learners' choice of representations (Weigand, 1991) Learners do different actions with different numbers of representations and different combinations of information that these representations hold (Ainsworth, Bibby and Wood, 2002) Differences in modelling when given text and when given graphics (Lohner, van Joolingen, and Savelsbergh, 2003)	Visualisation of graphs through automatically produced graphs (Elliot et al., 2000) Interpreting graphs through animation (Scanlon, 1998) Visualisation of graphs through drag and drop (Bodemer and Faust, 2006) Self constructed representations (Borba and Confrey, 1996, Cox, 1996) Visualisation through dynamic linking (Van der Meij and de Jong, 2006) Ineffectiveness of animation (Morrison and Tversky, 2001)	Preferences for graphs, numbers and equations (Keller and Hirsch, 1998) Approaches to graphs (Ruthven, 1990) Point-wise and Global approached to functions (Even, 1998) Algebraic and Visual approaches (Villarreal, 2000) Algebraic, numeric and graphic strategies (Senk and Thompson, 2006) Multiple strategy effect (Tabachneck et al., 1994) Three types of solution aids (Novick, 1990) Use of strategies with dynamic and static (Lewalter, 2003) Characterisations of strategies in visualising representations (DeWindt-King and Goldin, 2003; Yoon and Narayanan, 2004; Laeng and Teodorescu, 2002; Johansson et al., 2005; Goldin-Meadow et al., 2001) Examining strategies based on utterances, sketches and actions (Weigand and Weller, 2001) Strategy as a unit of analysis (Aczel, 1998)

Table 2-1 shows how the key studies reviewed in this chapter are organised. It presents a framework of the studies which are categorised into different themes. It gives a glimpse into the gaps in the literature.

This chapter begins by identifying frequent difficulties with standard external representations when instantiated in the context of learning. Various studies are examined, typically with a focus on mathematical representations. From these studies are drawn some theories intended to explain the reasons for the difficulties identified about learning with multiple representations. The chapter goes on to examine attempts to reduce these difficulties, including the design of computer environments, the provision of alternative instantiations and representations constructed by learners' themselves. Finally, a number of studies are revisited with a view to exploring the extent to which using "strategy" as a unit of analysis might be a valuable way of examining and explaining learners' engagement and difficulties with multiple computer based representations.

2.2 IDENTIFYING DIFFICULTIES IN LEARNING WITH MULTIPLE REPRESENTATIONS

For many years, researchers have been attempting to address some of the difficulties in learning with multiple representations. However, difficulties related to linking multiple representations in the context of learning still persist. It is important to establish causes for these difficulties. This section makes a case that there is no one explanation that covers the complexities of learning with multiple representations. It describes some of the various difficulties that learners appear to have with multiple representations and some of the related theories that may help explain the difficulties people encounter in learning with multiple representations.

2.2.1 DIFFICULTIES RELATED TO STANDARD EXTERNAL REPRESENTATIONS

This section focuses on different studies about learning with standard external mathematical representations.

An important insight into learners' use of mathematical representations is provided by Zazkis and Liljedahl (2002) is about learners' pattern-finding with numerical representations. A study about learners' attempts to generalise from numerical representations was conducted with 36 pre-service school teachers. Learners were given a task to find a pattern based on an array of numbers. Zazkis and Liljedahl wanted to find out, for example, whether their assumption that given an array of numbers, the expression of patterns would have been expressed in external representations using algebraic forms. The participants of this study were asked to log their thinking in a journal while completing the task; and not just give their final answer. The journal logs were analysed and it was found that the expression of patterns was not always accompanied by, and did not depend on, algebraic notation alone. Zazkis and Liljedahl associated this with the difficulty that the participants' experienced in performing operations involving algebraic symbolisation. This study also found that the participants perceived their final answer as inaccurate whenever it is not accompanied by algebraic symbolisation. The study shows that learners can experience difficulty related to operations of standard external representations and that this difficulty can be related to the way they represent and express the process of operation in tackling these representations.

Another study (Knuth, 2000) relates to a mathematical problem involving algebraic representations and corresponding graphical representations. Knuth conducted a study with 178 high school students taking college preparatory mathematics courses. His assumption was drawn from that of Schoenfeld et al. (1993) that

> *"Understanding both the equation-to-graph and graph-to-equation connections is considered fundamental in developing the flexibility to move fluently among representations, and the connections are often taken as mathematically straightforward." (p.501)*

The problems used in this study are about linear functions (e.g. Figure 2-1 where the graphical representation is a line corresponding to an equation written in algebraic notation). The representations are said to be not computationally equivalent because although they hold equivalent information, the computational requirements to manipulating them are different with each (see definition in section 1.2 from Larkin and Simon, 1987).

The problem task given can be answered either by scrutinising the line graph or by operating with the equations. The learners are asked to give their solutions and explanations to their answers to the problems. Knuth explains that even when a graphical solution method is more suitable, students tended to choose an algebraic method and use the former to support the latter method. In the analysis of solutions, according to Knuth, students find difficulty in establishing the connection between a point on the line graph and the name of that point given in numbers (known as 'numerical coordinates'). In fact, Knuth identifies that students normally assume that an exact solution is required, thus leading students towards an algebraic solution. He argues that

> *"students' reliance on algebraic solution is due to their failure to recognise the points used in calculating a 'slope' as solutions to an equation—recognition of which should make a graphical-solution method a viable option—rather than to a perceived need for precision." (p.505)*

The choice of algebraic representation, some may argue, may be attributed to didactic teaching of mathematics (c.f. Bloch, 2003) that is typically focused on algebraic symbolisation. However, the point being made here is that learning with multiple external mathematical representations can be problematic. Zazkis and Liljedalh's study shows that difficulty related to operations of one standard external representation (i.e. algebraic) can be related to the expression of this representation. The second study suggests that learners also experience difficulty relating numbers with graphs thus leading learners to choose, for example, algebraic representations. Other studies have confirmed these findings (e.g. Dunham and Osborne, 1991 below).

The graph below represents the equation $?x + 3y = -6$. (We do not know the value of the coefficient of x.)
(a) Is it possible to find a solution to the equation without the missing value? Explain your answer.
(b) How could we find the missing value? Explain your answer.

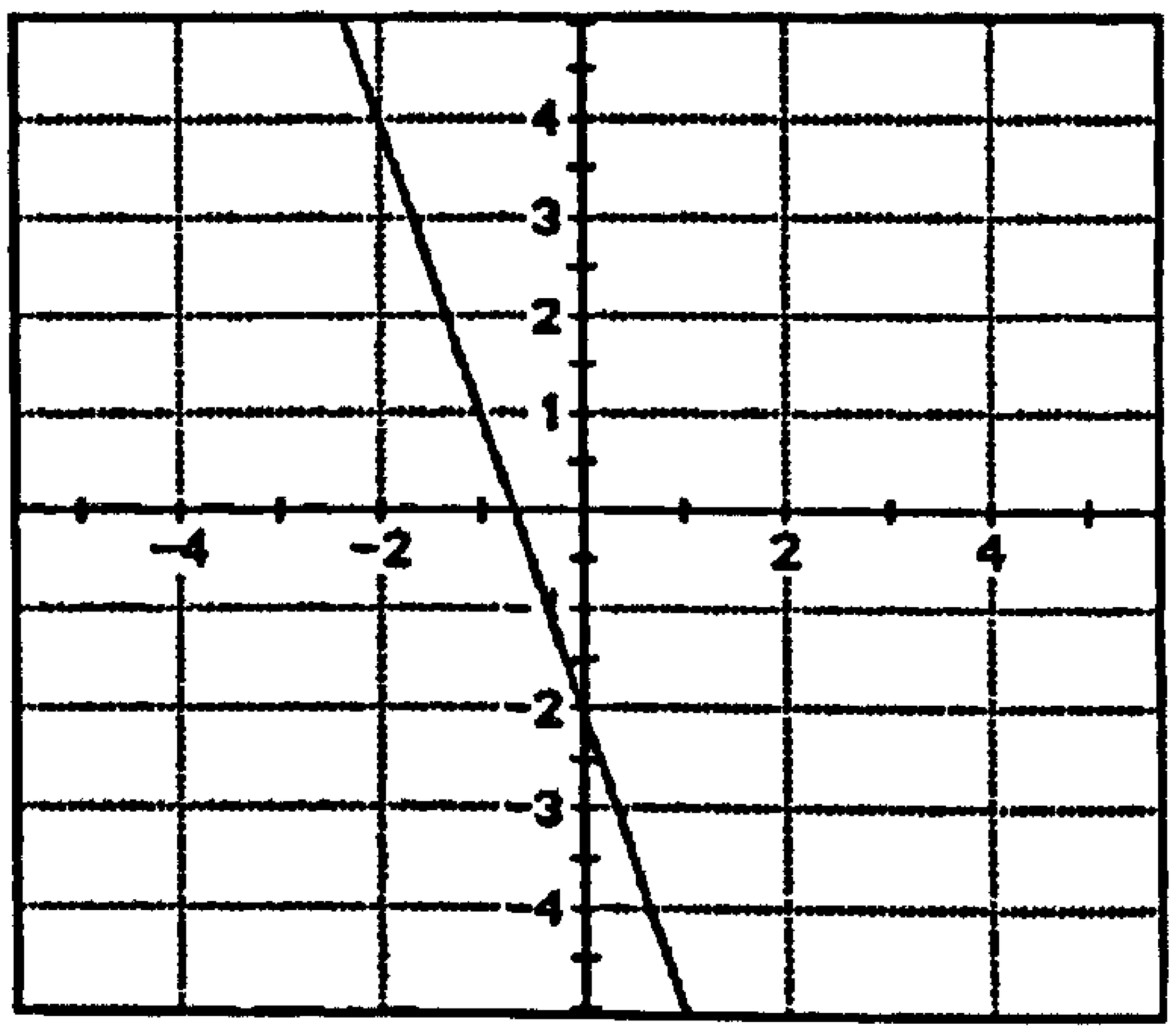


Figure 2-1 An example of a linear function problem (Knuth, 2000, p. 503)

These findings are corroborated by Dunham and Osborne (1991), who attempted to categorise some of the difficulties with mathematical graphs. Four hundred university students took an exam in mathematics at Ohio State University. The exam items consisted of questions about graphing algebraic inequalities e.g.:

> *"Solve the inequality $4 - x$ is greater than or equal to $(3x - 1)/2$. Use the graphs to solve $f(x) < g(x)$."*

Students' responses to the questions were analysed. Although the exam items (like the example above) included graphical and algebraic representations students' responses also included numeric representations. By analysing students' answers, five different types of difficulties associated with graphs were identified:

1) Extraction of numerical information based on graphical representation: learners have difficulty naming a graphical point with its corresponding numerical equivalent. This is similar to Knuth's finding above.

2) Failing to see mathematical graphs, for example a line, as an infinite collection of points.

3) Mathematical graphs have distinct points (i.e. points that are definite, e.g. a point which is an intersection of two graphs). The learners' tendency is to focus on distinct points and to ignore other points that are not distinct.

4) Mathematical graphs can be presented in different sizes depending on how they are scaled. Learners can be inattentive to the importance of scales.

5) Graphs can be transformed (i.e. a new created graph based on an existing graph) or re-scaled (i.e. zoomed-in and zoomed-out views of a graph). Learners can sometimes confuse transformation and rescaled views of a graph.

Considering the difficulties identified by the studies presented in this section, it is possible to group the problems learners experience with standard external mathematical representations (Table 2-2). The categories depend on the type of external mathematical representations learners are tackling.

Table 2-2 Difficulties with standard external representations

Numeric	Algebraic	Graphical	Combination of two or three types
• Pattern finding • Operations with numbers	• Expression of algebraic notation • Operation with algebraic notations	• Distinguishing: one graph to another, and one scaled graph to another rescaled graph • Inattention to importance of scales • Over focusing in distinct points	• Establishing connection between them • Translation of one representation to another • Emphasis of the use of one representation over the other

The difficulties in learning with standard external mathematical representations given in Table 2-2 show that the difficulties are distinguishable from each other according to the type of mathematical representation. There are many recent studies reporting learners' difficulties with these three mathematical representations (e.g. Duval, 2006; Warren and Cooper, 2008).

2.2.2 DIFFICULTIES RELATING TO COMPUTER-BASED REPRESENTATIONS

In the previous section, some of the difficulties in learning with external mathematical representations were categorised. Standard external representations can be presented in using different technologies. When external representations are computer-based, they can be presented differently. For example (as outlined is section 1.2 above) a graph can be given to learners in different designs such as i) graphs are non-moving - display is non-changing; or ii) elements of the graph can 'move' - display automatically changes e.g. imagine a point along a line graph moving up and down iii) elements of the graph can be manipulated – display changes by directly moving elements of a graph e.g. using a mouse to move elements around the screen. Each of these instantiations may introduce its own inherent difficulties. This section investigates the kinds of learner difficulties associated with each of these types of instantiation.

One study investigated learners tackling external representations about 'Chemical change': this involved graphs, pictures and symbols of chemical notations (Ardac and Akaygun, 2005). A piece of software about chemical change was designed. One group of learners was assigned to use this software which it allows them to view moving external presentations as a video clip. Screen shots of the same materials are reproduced and were given to another group of learners. The second group viewed the representations using an 'Overhead Projector'. One of the analyses Ardac and Akaygun conducted was on the learners' sketches of 'molecular change' before and after viewing the displays. In comparing learners' performance between the two conditions given, those given dynamic instantiation performed better than those given static representations. The reduced performance of those learners who used static instantiation was attributed by Ardac and Akaygun to a particular difficulty with processing information from a static display. They suggest that learners find it difficult to construct an image based on a static representation. This difficulty can therefore be characterised as relating to instantiation. Ardac and Akaygun explained that when learners are presented with static pictures showing the motion of particles, learners may attempt to make a mental image of moving particles. It is likely that learners cannot accurately make a mental image of this 'motion'. This gives a dynamic instantiation an advantage over a static one in this context, as the information processing required in constructing mental images may have been supported by the type of instantiation.

The next example of difficulty relates to manipulation of computer-based representations in terms of how elements of representations move about the computer screen (Lowe, 2003).24 undergraduates were given a weather map and asked to predict the pattern of meteorological markings that would occur in 24 hours time. Half the participants (the 'animation group') were given a computer-based interactive animated weather-map sequence to help them; the other half control group were not. The animation group could control the speed of animation. Both groups were asked to draw their predictions on a blank map. The researcher analysed video-records together with the drawings and written descriptions of how various patterns of weather change over time. According to Lowe learners in the animation group seemed to extract information on the less important concepts due to the manipulation involved in animating representations. Lowe further concluded that there are some learners that are having difficulty in extracting information in complex animations because the learners focus on one part of a display at a time. Lowe believed that full attention to one part of the display would result in disregard of other parts of the display. While this study shows that animating external representations can introduce difficulty due to the demands of manipulation, it also provides an account that this difficulty is linked with focus of attention.

While Ardac and Akaygun found no difficulty concerning the use of dynamic instantiation, Lowe on the other hand found that learners have difficulty with moving displays. These two studies therefore show that dynamic instantiation may have different effects on learning seemingly dependent on the levels of control over speed of animation.

Another study, Jones and Scaife (2000), corroborates this difficulty in focusing attention when using animation. They investigated two groups of students learning about the heart. One group was given an animation; the other was given three printed diagrams. 112 learners, aged 13 and 14, participated in this study. Two different task types were used in the study: i) open task – e.g. finding out how the blood flows though the heart ii) structured task – e.g. investigating certain aspects of the heart using a worksheet provided. A post test was given to both groups which asked learners to draw arrows to complete the 'human heart blood-flow' diagram (Figure 2-2). Learners' utterances elicited during the engagement with the task were also analysed. In terms of comparing performance between the two groups, Jones and Scaife reported no significant differences in test performance in completing the blood flow diagram between the animation and static groups. However, they found that their participants performed better using a structured worksheet than using an open task regardless of the instantiation. Jones and Scaife argue that the difference may be due to the organisation of information presented in the worksheet used; and believe that the order in which the information is presented facilitates certain aspects of the representations to which learners pay attention. They suggest that if the animation group were able to control which part of the screen animates, this group might perform better than the static group.

Otero, Rogers and du Boulay (2000), however, show that when learners manipulate elements of the screen in order to control how these elements move about, other kinds of difficulty are introduced.

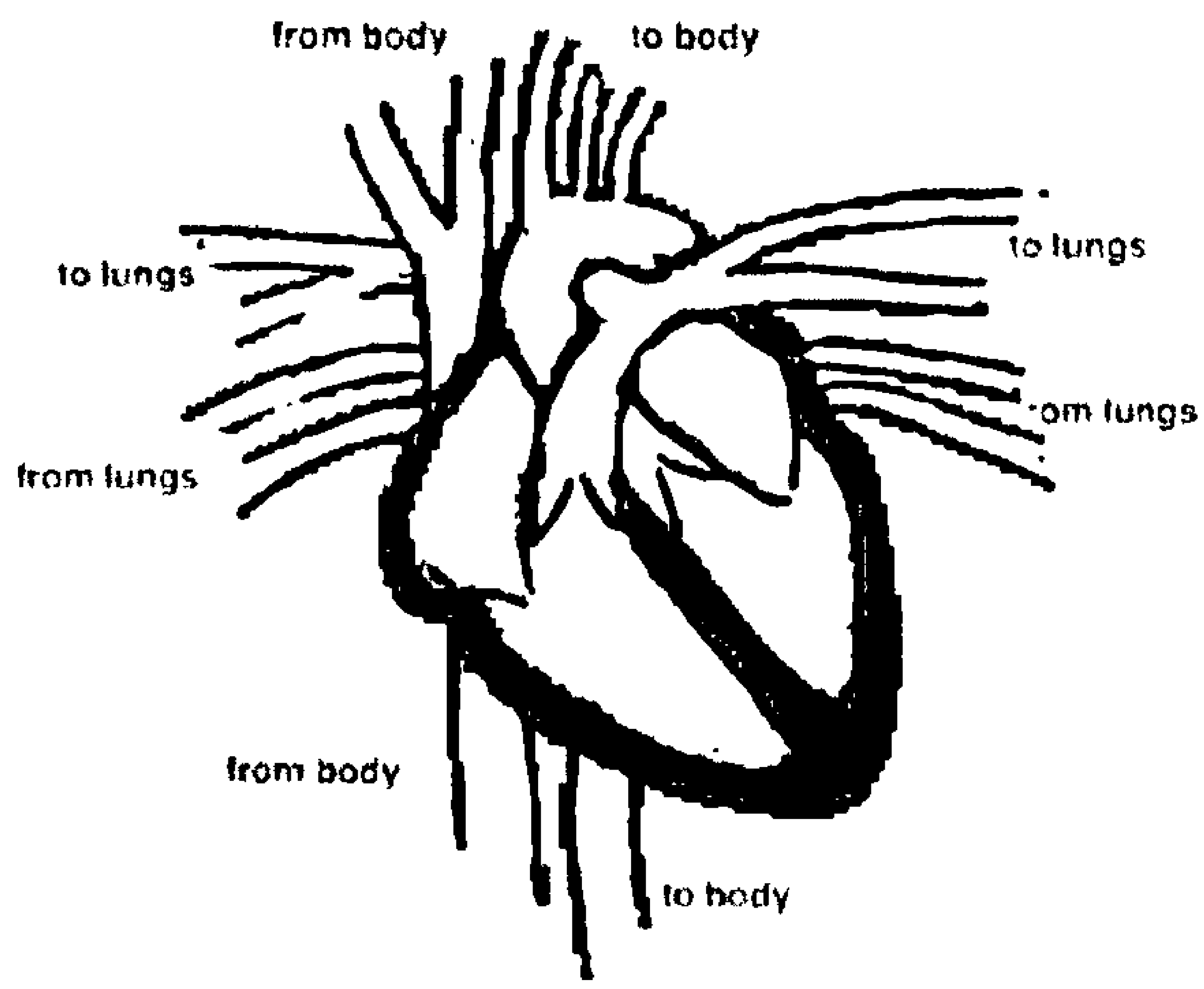


Figure 2-2 The test diagram in the study of Jones and Scaife (2000, p. 237)

This study relates to moving certain parts of the display by using a computer mouse. Eighty university undergraduates consisting of mathematics and non-mathematics students were randomly assigned to four different conditions (Otero, Rogers and du Boulay, 2001). Two kinds of external representations were used 2D and 3D graphs. The four conditions were made up of two systems on 2D representations and two other systems on 3D representations. The systems varied in the degree of manipulability of the elements of the graphs, one having fewer manipulable elements than the other one. . The task given was a geometry problem about stereographic projection involving the processing of angular relationships between objects. Otero, Rogers and du Boulay hypothesised that the more learners can manipulate elements of the graphical representations the more they can grasp geometry concepts involved in the task. The participants' computer screens during the task completion were video recorded. The participants were pre and post tested on their spatial abilities and geometry knowledge. In testing the effect of the degree of interactivity based on the geometry test, the results showed no difference in all four conditions. The researchers also concluded that adding manipulability does not assure learning gains for all the participants. One of the suggestions of Otero et al. was that the manipulability of graphical representations could introduce complexity such as learners focusing too much on the graphical representations and paying less attention to other important information offered on the screen. However, Otero et al. pointed out that the design of instantiation, for example, revealing important information relevant to an element being manipulated may help direct learners' attention to other parts of the screen and help in translation of external representations. It is getting clearer that the difficulty relating to manipulation of computer-based representations can also be associated with the representations to which people pay attention.

In Table 2.2, difficulties were grouped according to the type of representation. In this section, the studies give some account of difficulties that can be grouped according to the type of instantiation (Table 2-3).

Table 2-3 Difficulties relating to instantiations

Static display	'Animated'	Draggable objects	Combination of two or three types
• Construction of multiple images to create a moving image mentally	• Extracting information • Focusing attention to certain parts of a display	• Focusing on manipulable elements	• Handling attentional resources

Sections, 2.2.1 and 2.2.2 have suggested that learners encounter difficulties with external representations and instantiations, respectively. The next section looks into possible theories that may help provide some of the reasons for these difficulties.

2.2.3 THEORIES RELATING TO DIFFICULTIES WITH MULTIPLE REPRESENTATIONS

There are theories that may help explain why learners have the particular difficulties, identified in sections 2.2.1 and 2.2.2, with external representations and instantiations. This section distinguishes some of the theoretical explanations for these difficulties, depending on whether the causes of difficulties of learning with computer-based representations are attributed to cognitive load or interactivity. The issues related to these theories are presented in this section in order to provide a rationale for examining learners' strategies with multiple computer-based representations.

The framework proposed by Rogers and Scaife (1996) is widely recognised in the research about graphical representations. The framework is based on literature relating to computer-based representations that emerged before 1996. The following aspects are relevant to why different external representations and instantiations in some cases may help but in other cases may hinder learning with multiple computer-based representations

1. *Computational offloading*: the extent to which different external representations reduce processing required to solve informationally equivalent problems.

2. *Re-representation*: the way different external representations having similar structure make problem solving easier or difficult.

3. *Graphical constraining*: the influence of graphical representations constraining the kind of inference that can be made about the underlying represented world.

In relation to the task of explaining why difficulties occur with external representations and instantiations, cognitive load theory appears to hold out the potential of a more precise model of how different factors interact to influence outcomes in a particular context.

Cognitive load theory is about the capacity of an individual to hold information in his or her working memory, for example during problem solving (Sweller and Chandler, 1991). Working memory is the limit of information that can be stored and introspectively processed by an individual within a give amount of duration (Chandler and Sweller, 1996). This theory provides some explanations as to how single between external representations and interacting with these external representations occur when solving a problem in the context of learning (Chandler and Sweller, 1991). Chandler and Sweller differentiated three types of cognitive load, and two of which are relevant to the difficulties identified in sections 2.2.1 and 2.2.2. The first is 'intrinsic cognitive load' which, according to Chandler and Sweller, is associated with the inherent difficulties relating to the external representations. For example, difficulty calculating numbers can be different from difficulty performing operations on algebraic symbolisations. The second type of cognitive load is 'extraneous cognitive load', also referred to as the 'unnecessary cognitive load' – the processing required when e.g. interacting with the representations and the way representations are presented. For example, in comparing many graphs, there are different difficulties in comparing graphs when one sketches a number of graphs than when one uses a graphing technology to automatically produce many graphs.

To further illustrate cognitive load theory and its relation to difficulties with different external representations and difficulties with different kinds of instantiations, two kinds of effects associated with cognitive load are given in the two studies below (Mayer and Moreno, 1998; and Kalyuga, Chandler and Sweller, 1999). These effects are:

> Split-attention effect – when attention paid to different representations presented simultaneously increases cognitive processing (see Chandler and Sweller for a detailed definition).
>
> Redundancy effect – when different representations presents the same information, additional cognitive load is not due to split-attention but due to the presence of multiple sources of similar information (see Kalyuga, Chandler and Sweller for more explanations).

Mayer and Moreno (1998) draw attention to split-attention effects, "in which attention to one type of presentation component may result in information being missed in a different accompanying presentation component" (p.158). This is similar to the claim by Mayer and Anderson (1992), focused on pictures, that such limited attention may hinder the connection of one part of a display to the rest. One of the experiments that Mayer and Moreno conducted included watching an animation of lightning formation. Seventy-eight students were divided into two groups, and assigned to either condition: animation with concurrent audio narration and animation with concurrent text. Two of the tests given were recalling relevant steps in lightning formation and correctly labelling elements in an illustration of lightning formation. Mayer and Moreno hypothesised that no difference in test results should exist between the two groups, because participants receive the same information. However, they contended that the cognitive processing required to watch animation while at the same time reading text may be different from the cognitive processing of watching animation whilst listening to audio. This difference according to Mayer and Moreno can be considered a split-attention effect. The results show that students who received animation with narration performed better than the text group in recalling information. Also, in labelling elements in an illustration, the narration group outperformed the text group. Mayer and Moreno suggested that the text group performed less well because of the cognitive demand in visual working-memory load in processing both the pictures and the text together; and that the audio group used two different cognitive resources, i.e. visual for processing the animation and auditory for processing the narration. Mayer and Moreno give an example of a change in modality of a certain representation changing learning performance.

Some studies have investigated redundancy effect according to the number of representations given. Kalyuga et al. (1999), reporting two different studies, suggest that when a learner is able to do a task with a single representation (e.g. visual such as text), adding another representation (e.g. auditory) increases cognitive load. They designed interactive self-paced computer-based material consisting of diagrams and text. They varied the design of the material by various means giving learners only text; or text with audio (like text as 'subtitles' to audio – what is said out loud is also presented as text at the same time); or audio alone. They found differences between groups of learners' understanding of 'fusion diagrams'. They tested learners by giving them a faulty fusion diagram and learners were asked to identify the faults.

The results show

> *"Auditory presentation of text proved superior to visual-only presentation but not when the text was presented in both auditory and visual forms. In that case, the visual form was redundant and imposed a cognitive load that interfered with learning." (p. 351)*

Kalyuga et al. suggest that multiple external representations can impede learning, especially when a single representation is sufficient for understanding a certain concept.

However, in this study, it was impossible to detect which representation is being considered by those learners given audio and text. It may be possible that learners are considering just audio while ignoring the text or possibly considering just text while ignoring the audio.

Although the two studies presented above are in the context of different representations available either as auditory and visual, it presented some evidence of how processing two types of information requiring different cognitive effort can lead to difficulty in handling information processing. When the design of the environment includes different external representations, a redundancy effect may occur. However, a split attention effect can also occur in the same instance. Examining effects based on tests can be complicated when instantiation can also introduce split-attention effects.

Table 2-4 Relationship of theories to difficulties

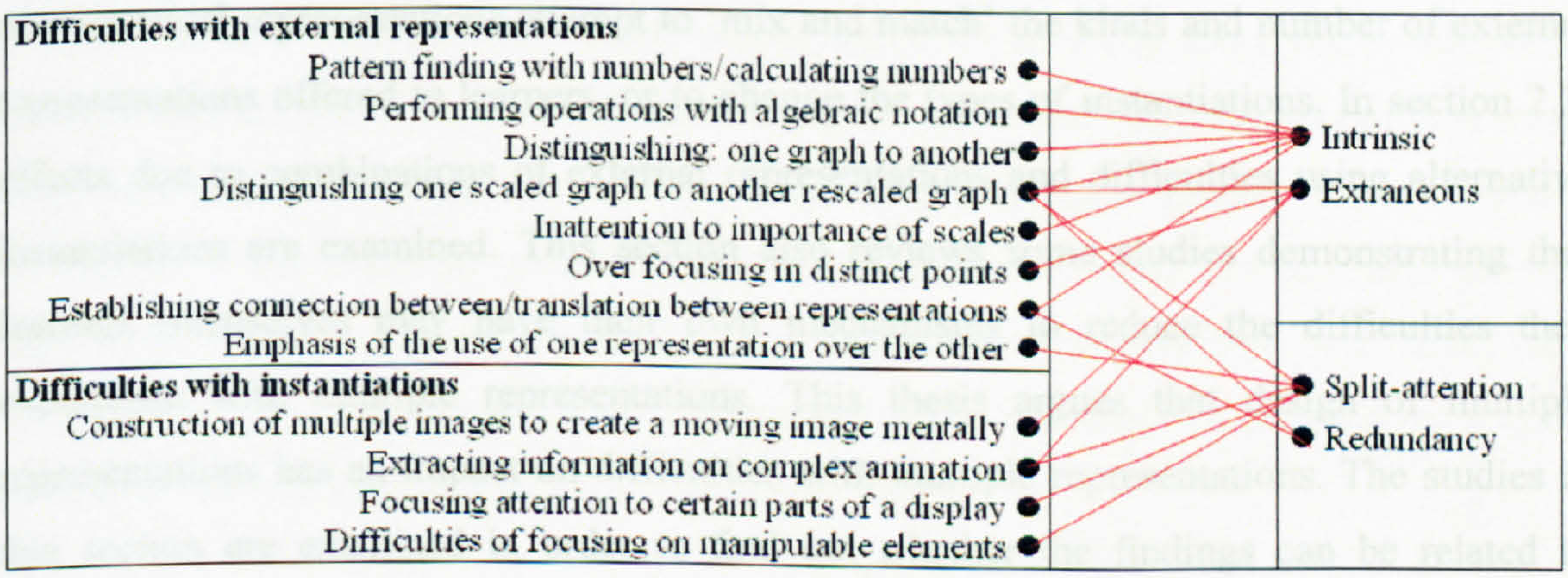


Table 2-4 is provided to show the relationship between cognitive load theory and difficulties with multiple representations. It shows that there are difficulties related to external representations or to instantiations that can correspond to both intrinsic and extraneous cognitive load; or to split-attention, redundancy. It demonstrates the complexities of explaining the reasons for difficulties of learning with multiple representations. For example, the difficulty "distinguishing one rescaled graph to another rescaled graph" can be identified as either an intrinsic cognitive load or an extraneous cognitive load. It could also be an effect due to split-attention or redundancy effect.

So, although the specific associations between multiple representations and learning have not been established empirically, researchers appear to be in broad agreement that the influence of representations on learning is a complex mix of form, function, interaction — and context. In order to inform educational design, we need a more precise model of how different factors interact to influence outcomes in a particular context. This requires a more precise understanding of the role each of the factors (i.e. representation, instantiation, strategy as discussed in this thesis) plays. It is suggested that the extent to which these explanations are interconnected should not be overlooked.

2.3 REDUCING DIFFICULTIES IN LEARNING WITH MULTIPLE REPRESENTATIONS

There are some ways to address difficulties when learning with computer-based representations. Some learning resources, for example, may carry with them some assumptions of ways to reduce these difficulties. Those who design learning with mathematical representations attempt to 'mix and match' the kinds and number of external representations offered to learners, or to change the types of instantiations. In section 2.3, effects due to combinations of external representations and difficulties using alternative instantiations are examined. This section also reviews some studies demonstrating that learners themselves may have their own mechanisms to reduce the difficulties they experience with multiple representations. This thesis argues that design of multiple representations has an impact on difficulties with multiple representations. The studies in this section are examined in order to find out whether the findings can be related to influences on strategies.

2.3.1 REDUCING DIFFICULTIES USING COMBINATIONS OF REPRESENTATIONS

Three empirical studies are presented in this section about the effects of external representations given to learners. The studies illustrated how learners use the representations offered to them on the screen. This section emphasises the significance of the choices of combinations of representations when learners use them.

In a mathematical task, 'Properties of Mathematical Iteration Sequences', Weigand (1991) investigated how computer-based representations may reduce learning difficulties with this topic. Weigand designed a piece of computer-based software in order to investigate which representation can best help learners describe the properties of iterations. There have been some claims (e.g. Sierpinska, 1994) that learners find iteration sequences difficult to tackle. The assumption was that different choices of external representation can have different influences on students' attempts to describe iterations' properties. There were seventy-nine learner-participants (11th graders) who were asked to explain the behaviour of an iteration sequence based on their choices from representations given by the software used in the study. The software generates an output representation that learners choose from: numbers, graphs, or equations. To illustrate the software, learners were given an initial equation. So, in Figure 2-3 for example, the equation is $y = 0.5x + 1$. Then they were expected to input a starting numerical value for the iteration sequence. So in Figure 2-3 the learner's initial numerical input was -4.5. Then the learner was expected to try out each of the different choices of representations. The left side of Figure 2-3 shows the output if choice '6 Graph' is selected. The learners were asked to complete a 10-item test about the properties of iteration and the use of representations in completing the task. The analysis of data show that when learners choose graphical representations (i.e. choices: 'graph', 'arrow-diagram' or 'cobweb-diagram'), they generally gave pictorial descriptions. The data also showed that when learners choose numerical representations (i.e. choices '4' 'table' and '1''beginning of the sequence'), their explanations were related to numbers and not graphs. Examples taken from Weigand's study are given below.

> *Choice 1 'beginning of the sequence': "Because k increases by 1, a_k by 1.5 and k begins with a higher value than a_k, the k values will be overtaken by the a_k values after 11 terms."*
>
> *Choice 6 'Graph': "After a while, the graph becomes parallel to the x axis"*

Weigand found that the kinds of external representations that learners choose influence the kinds of interpretations they make. It also shows that talk data can be linked to a particular kind of external mathematical representation and can be related to the kinds of representations that learners considered. While this study shows that the extent to which the choice of standard external representations, as an output representation, influences the description of the properties of iteration, this study did not explain how the different representations that can be generated by the software in a given time influences the kinds of interpretations that learners make. Since the software can generate many different representations in a short time, the many representations produced demand cognitive effort which learners have difficulty grasping. The attempt to make graphical representations computer-based in order to help learners describe properties of iteration may fail because according to Weigand learners may develop incorrect concepts with the many visual representations generated automatically by computers.

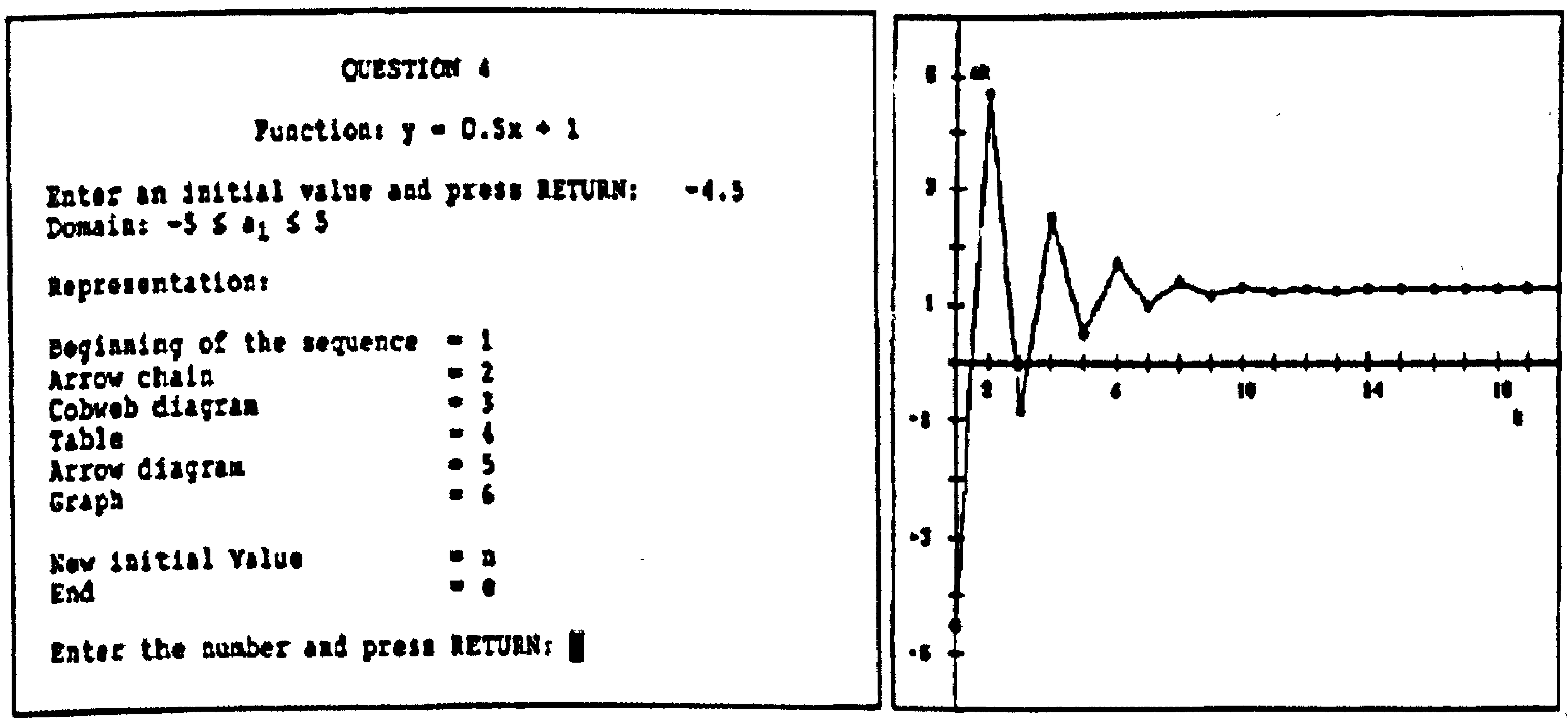


Figure 2-3 Example of external representations in Weigand's study (1991)

Ainsworth (2006) points out a difference between a study she and her colleagues conducted in 2002, and a study by Kalyuga and others in 1999 (see section 2.2.3 above). She suggests that while providing more than one representation may have a negative effect on learning when both representations provide exactly the same information, multiple representations can be beneficial when information that each representation provides is different to each other. Ainsworth, Bibby and Wood looked at learners' performance on computational estimation tasks. The task was about estimation. Ainsworth, Bibby and Wood designed three programs. The number of external representations was made constant. That is, each program used two kinds of external representations (see e.g. Figure

2-4). Figure 2-4 shows the three programs used in the study. The first program used pictorial representation (top-figure). The second program used mathematical representations (middle-figure), and the third program used a combination of pictorial and mathematical representations (bottom-figure). The designs of the software were varied in terms of the combination of the representation and the dimension of information this representation contains. For example, the left of the middle-figure of Figure 2-4 displays two dimensions of information, i.e. magnitude of the number and the direction (e.g. the negative sign meaning a downward direction). In one of the experiments conducted, forty-eight Year 5 students (age range 9 to 11 years old) took part. A pen and paper test about estimation was given to measure performance of students before and after the use of a computer program. The students were divided into four groups. Three groups were assigned to each program. The fourth group did not use a computer program and only took the pre and post tests. The participants' scores on the pre-test did not differ between the four groups but scores in the post tests differed between groups. The post-test scores of those who used programs 1 (pictorial) and 2 (mathematical) improved compared with the pre-tests scores but there was no improvement in the scores of the control group and those who used program 3 (mixed of pictorial and mathematical). It seemed that all children learned estimating through the program.

The study of Ainsworth, Bibby and Wood and that of Kalyuga et al. show that the combinations of representations can affect learning. It seems that learners do different actions with different numbers of representations and different combinations of information that these representations hold. Although these studies did not attempt to reduce difficulties that learners experience with external representations, they show how design in terms of the number and choice of external representations can influence learning. So if learners tackle only one particular standard form of representation, their experience of learning may be different when they are tackling more than one form. This is confirmed by the next study by Lohner, van Joolingen and Savelsbergh (2003).

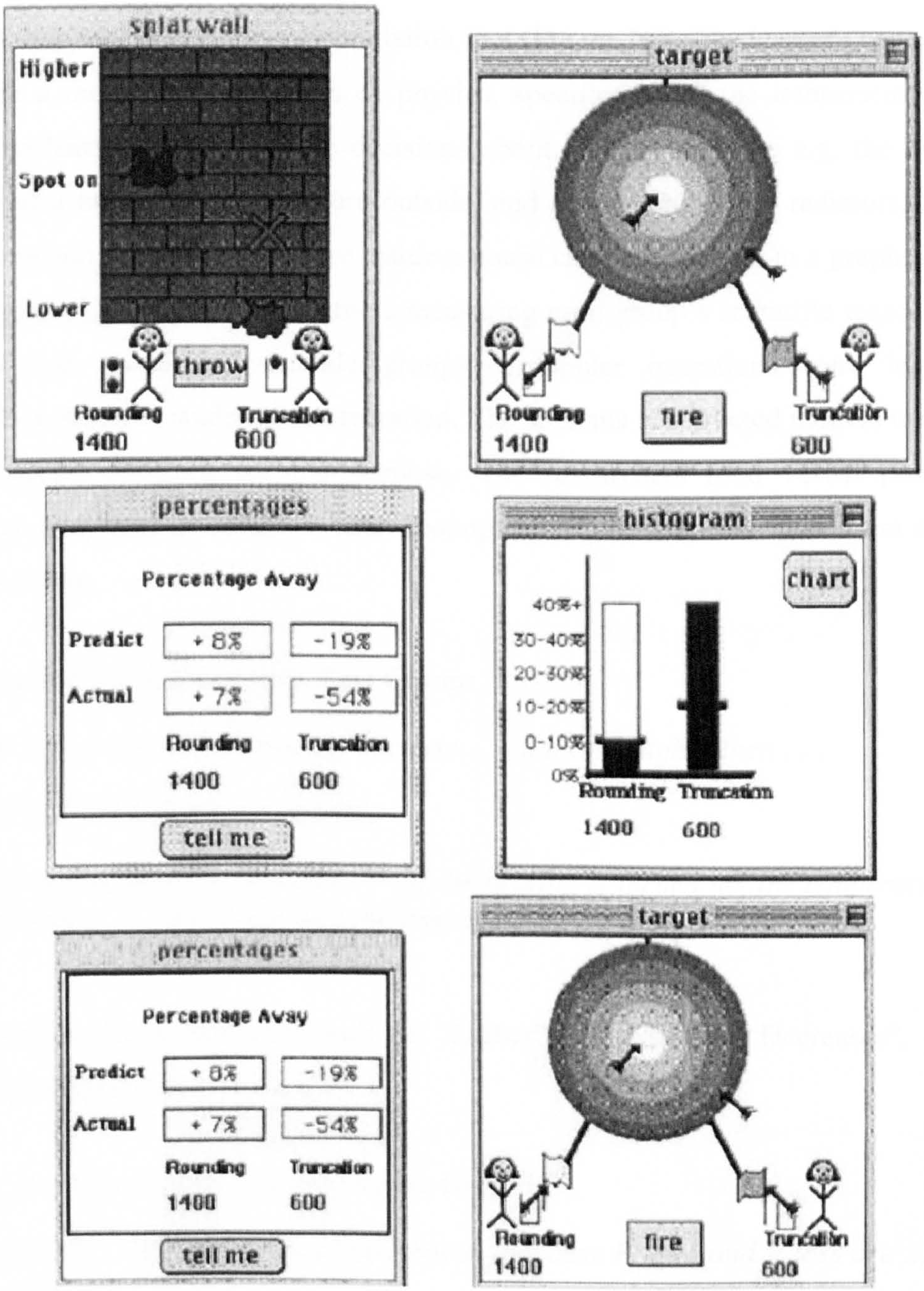


Figure 2-4 Example of representations used in Ainsworth, Bibby and Wood (2002)

Lohner, van Joolingen and Savelsbergh (2003) studied pairs of learners (forty-two secondary school students, with an average age of 17) divided into two groups: one working with text-based representations (the 'Text Group') and the other with graphical representations (the 'Graphic Group'). Lohner et al. investigated how learners construct computer models based on the representations offered – a model that fits data on either a text modelling tool or a graphical simulation tool (Figure 2-5). The learners were expected to provide a model in the domain of physics, specifically, of the temperature inside a house. The learners need to make decisions about several variables e.g. the number of windows of a house, the temperature outside, and power emitted by radiators, before an output simulation of the temperature inside a house can be displayed in a graph. Lohner et al. administered pre-tests to the learners measuring each group's scientific reasoning skills (between text group and graphic group). Computer operations were logged and conversation between students was recorded. The students constructed models were scored using a rubric designed for the purpose. The researchers used verbal protocols to triangulate with their quantitative data. Examples of conversations taken from this study are given below.

A and B participants from Text Group:

B: Let's start with P heating, because . . . if that is higher then . . .

A: Huh? But that's not possible

B: So you write for instance . . . if the heating is turned up, the temperature inside increases . . . if it's low then the temperature decreases

The utterances from this pair such as "higher", "Increases", "Decreases", indicated numerical interpretations.

C and D participants from Graphic Group:

C: And ... Let me see, I can see that in any case P total and P loss are ... together ... they are P heating

D: Yes but how do you want to? we can only make two-links cant we? Positive or negative influence?

(Ten minutes later)

C: No ... yes ... I don't know. Anyways I know that P total and P loss are together equal to P heating

(Four minutes later)

D: Err, there is something else I just saw. Oh yeah, I think that P total I is equal to P heating minus P loss

The data above showed that the pair of students in the Graphic group came up with precise equations.

Evidence from the computer logs suggests that the two groups engaged in different kinds of modelling between the two groups and based on utterances that students' scientific reasoning was different between different representations. In this study, the effect on computer models varied between text and graphics. For example, those learners involved with graphics made more use of a graph as a representation that extends information they hold in their head, than those of the text group, as evident from the conversations between pairs of learners.

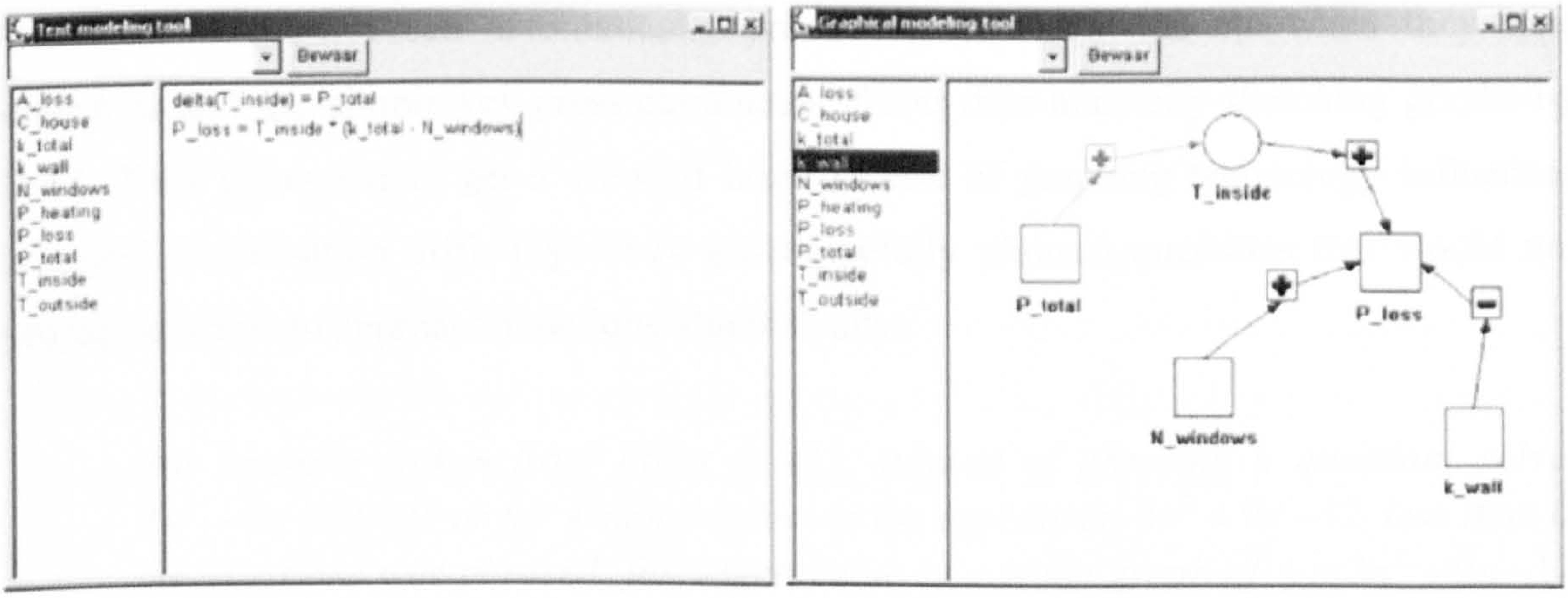


Figure 2-5 Examples of representations in the study of Lohner et al. (2003)

The studies examined in this section show that there are certain differences in learning with different combinations of representations. It also shows that the combination and representations given appear to influence learners' choice of representations and the way they use representations. However studies to date have not looked systematically at how learners respond to different ways of instantiating multiple mathematical representations (numeric, graphical, algebraic).

2.3.2 REDUCING DIFFICULTIES USING ALTERNATIVE INSTANTIATIONS

The studies here attributed their successes and failures of attempts in reducing learning difficulties to the kinds of instantiations.

Elliot et al. (2000) explain how some difficulties with visualisation of graphs are reduced through the instantiations offered by graphic calculators. The researchers reported the perceptions of six Year 12 students (age 17) on the role of visualisation in their understanding of 'functions'. They used questionnaires regarding visualisation; interviews about how the participants of the study would solve problems involving functions; learners' worked examples; and questionnaires concerning the role of technology. Graphical calculators can automatically produce graphs for students when they type equations and use simple key press commands, faster than manually sketching graphs by hand. They attempted to get a view of how the use of graphing technology influences learners' visualisation difficulty. They gave carefully phrased questions that would not prompt students to approach functions algebraically.

> *For example (taken from Elliot et al.), instead of phrasing a question: solve:* $3x^2 + 9x - 12 < 0$ *or for which x values is the expression* $3x^2 + 9x - 12$ *less than 0, the questions was phrased: for which value of x is the graph of* $y = 3x^2 + 9x - 12$ *below the x axis?*

The questions included were about graphing functions, exploring the effects of transformations of functions, finding inverse functions, solving equations, and investigating trigonometric and logarithmic functions. One of the results they found was that five out of six students would initially visualise a graph mentally which students found difficult (see e.g. extract below).

> *Student X found visualisation particularly difficult. "When I am not used to the type of problem, it is not easy to relate it to the graph or system, so I would use algebra."*

The extract suggested that these students would typically dismiss using a graphic approach. The use of graphing technology helped students with this kind of situation. Elliot et al. reported that their interviews with the students showed that the use of graphing technology helped students to formulate ideas about functions which in turn helped the learners developed insights about the problems given. Elliot and her colleagues claimed that the ease of accessing automatically-generated graphical representations via graphical calculators has a positive impact on learners' visualisation difficulties. They suggest that because automatically-generating graphs reduce difficulties, learners are more likely to choose to work with graphs. Elliot et al. also reported that learners favoured working with symbolic representations rather than visual when generating graphs using calculators because learners view visual forms as error-prone and inefficient (c.f. Weigand, 1991; Knuth 2000). According to Elliot and others, when the students made use of graphing technology, learners tended to combine algebraic and graphic representations often rather than concentrating on one representation. This study suggests that there is an interaction between the choice of representations and how these representations are presented to the learners. However, as with Weigand (1991), this study is not successful in picking out the influences of choice of representations and the features of graphing technology that helped the learners. In fact, while Elliot et al. claim that speed of automatically generating an output representation can help learners, Weigand suggests that this may depend on how many representations are being generated in a short time. If this assumption is true, it may be important to investigate what learners do with each of the many output representations that can be generated.

The different claims above pertain to one of the perceived affordances of automatically generating representations: immediate variation of the representation in accordance with learners' inputs. The studies by Weigand (1991) and Elliot et al. (2000) present two opposing views. Weigand suggests that automatically-generated graphical representations hinder mathematical thinking because learners tend to view graphs as a picture. As others suggest (see e.g. Even, (1998), discussed later in section 2.4.1) it can be helpful for learners to translate between graphs and equations to have a better understanding of the abstract mathematical concepts learners are tackling. Viewing graphs only as a picture means these links between graphs and algebra are lost. Elliot et al., on the other hand, believe that automatic generation of graphs can help with visualisation. These two points are difficult to bring together. It is not clear from the above studies whether the claims are related to the nature of the standard external representation (and to the particular choice of representations juxtaposed), or to the type of instantiation.

Another study that attempted to reduce difficulty with interpreting graphs is by making graphical representations animated. Scanlon (1998) examined students' use of algebraic and graphical representations in physics (specifically, some 'kinematics' and 'dynamics' problems). Data were taken from: a main study of individuals' and pairs' of students transcribed utterances and paper and pencil worksheets; an instructional case study of video-based observations and self-reports; and a classroom-based study of interviews, video and audio records of observation, students' written homework and worksheets, and observers' notes. Thirty-five students participated in the main study; a teacher and twelve students the case study; and a teacher and twenty-nine children in the third study, respectively. Further data based on a questionnaire were collected in the second study. In the main study, the task prompted students to use algebraic and graphic representations and to express on paper the relationship between the two representations. Scanlon identified that students experience difficulties in interpreting graphs. She stated that making representations dynamic using simulations, for example, as a feature could help students interpret graphs better. Scanlon suggests that instructional designers should carefully consider incorporating features of computer-based representations.

Scanlon, Weigand, Elliot and others aim to show how different instantiations can reduce difficulties with external representations. However, the examination of these studies suggests that the combinations of varying external graph-representations and the different ways of presenting graphs, either by animating or non-animating or by automatic generation of graph or not, may impact on the way learners solve problems with mathematical representations. The complexities of examining these kinds of learning interactions must be considered.

Scaife and Rogers (1996) and Ainsworth (2006) suggest that dynamic-linking between representations may provide further benefits to learners, for example by helping reduce visualisation difficulties. Dynamic-linking is the process of linking and manipulating representations at the interface which helps learners to visualise. Dynamic-linking happens when "learners act on one representation and see the effect on another one" (Ainsworth, 2006, p. 194). A study compared learning outcomes by changing the way representations are integrated and linked (Van der Meij and de Jong, 2006). For example, a 'line graph' has a 'graphical point' that is 'draggable' along it, and the position of this graphical point on the 'line graph' corresponds to some numerical values. Representations are linked, for example, when a person moved the graphical point and the corresponding numeric display changes. The numeric display can be positioned separately away from the linked graph. Representations are integrated, when, for example, the graph and the numbers are physically displayed together and not separately far apart. Van der Meij and de Jong (2006) investigated the differences of three learning environments (Figure 2-6):

1) with separate, non-linked representations

2) with separate, dynamically linked representations

3) with integrated, dynamically-linked representations

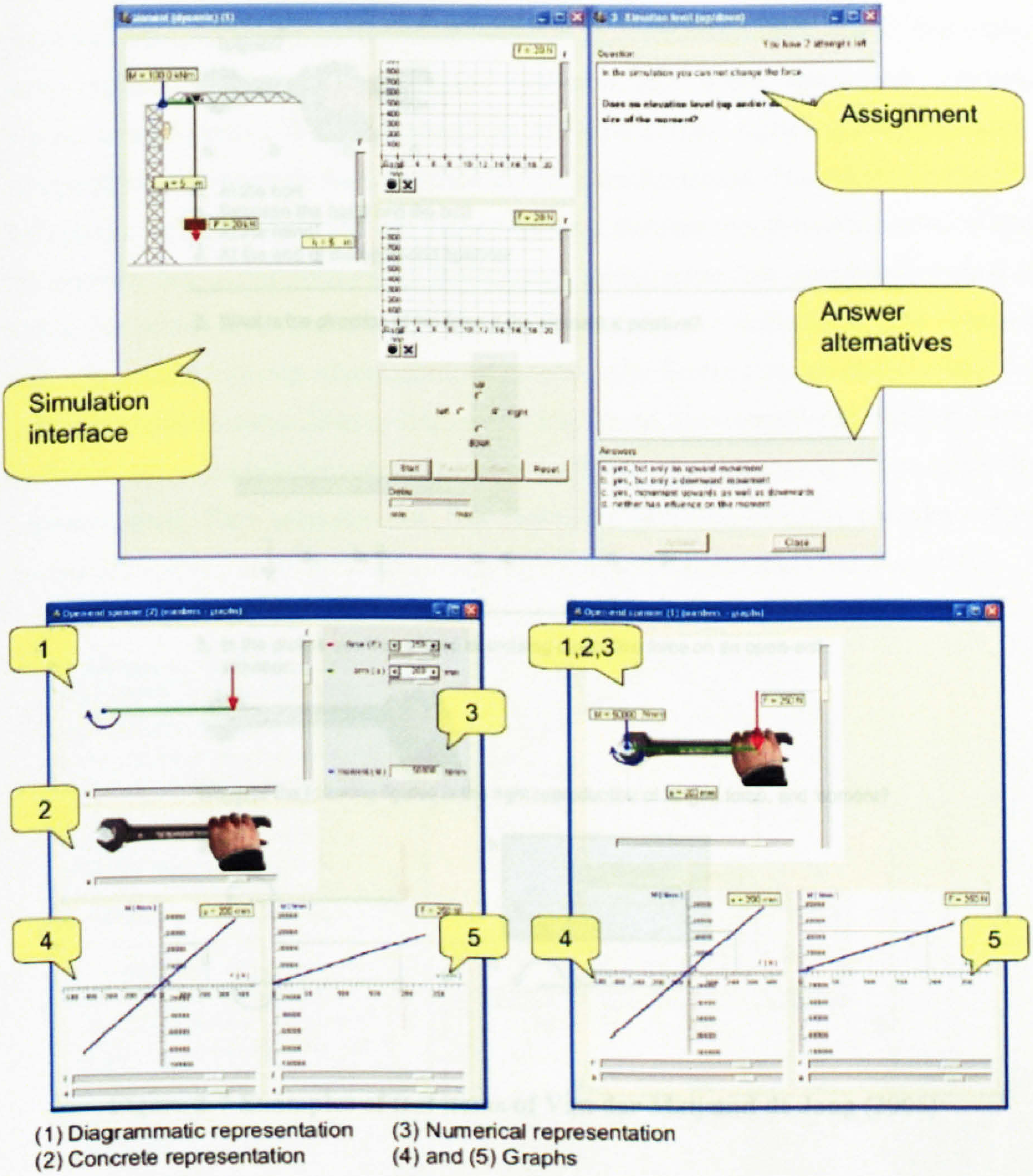


Figure 2-6 Examples of representations in the study of van der Meij and de Jong (2006)

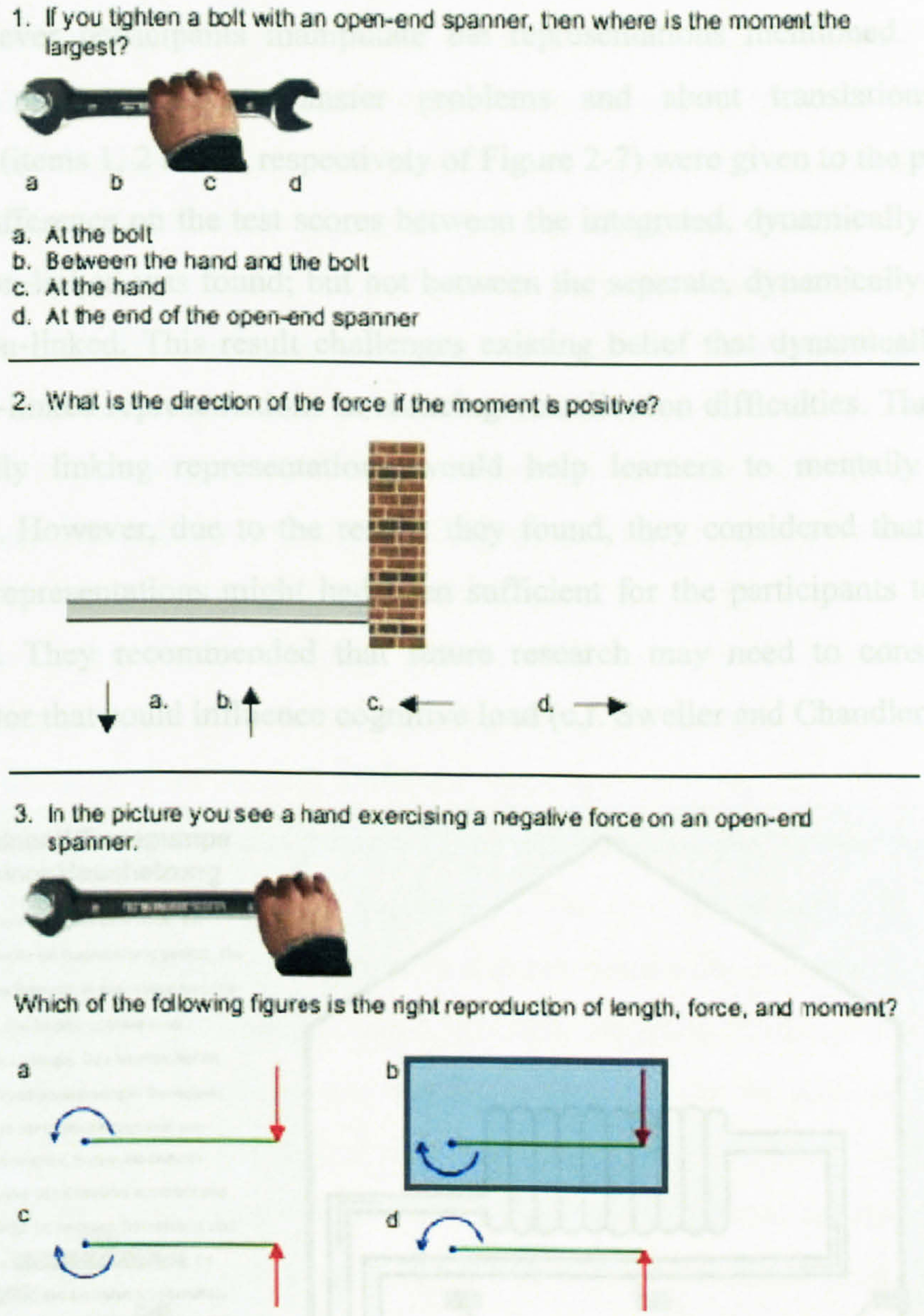


Figure 2-7 Examples of test items of Van der Meij and de Jong (2006)

Their expectation was that integrating and dynamically linking representations (i.e. learning environments 2 and 3) has an effect on learning. Seventy-two students (between 16 and 18 years old) were divided into three groups and were randomly assigned to conditions 1), 2) or 3) above. They analysed work with a computer-based simulation model about a Physics topic, 'Moments'. Their computer-based material engaged their participants to interact directly and manipulate some 'concrete' and diagrammatic representations using 'draggable sliders' (see Figure 2-6). The participants could also interact with numerical representations using 'arrow keys'. The changes in values occur in real-time whenever participants manipulate the representations mentioned. Test items about subject matter content, transfer problems and about translations between representations (items 1, 2 and 3, respectively of Figure 2-7) were given to the participants. A significant difference on the test scores between the integrated, dynamically linked and the separate, non-linked was found; but not between the separate, dynamically-linked and the separate non-linked. This result challenges existing belief that dynamically-linked is better than non-linked representations in reducing visualisation difficulties. They assumed that dynamically linking representations would help learners to mentally relate the representations. However, due to the results they found, they considered that the colour coding of the representations might had been sufficient for the participants to relate the representations. They recommended that future research may need to consider colour coding as a factor that could influence cognitive load (c.f. Sweller and Chandler, 1991).

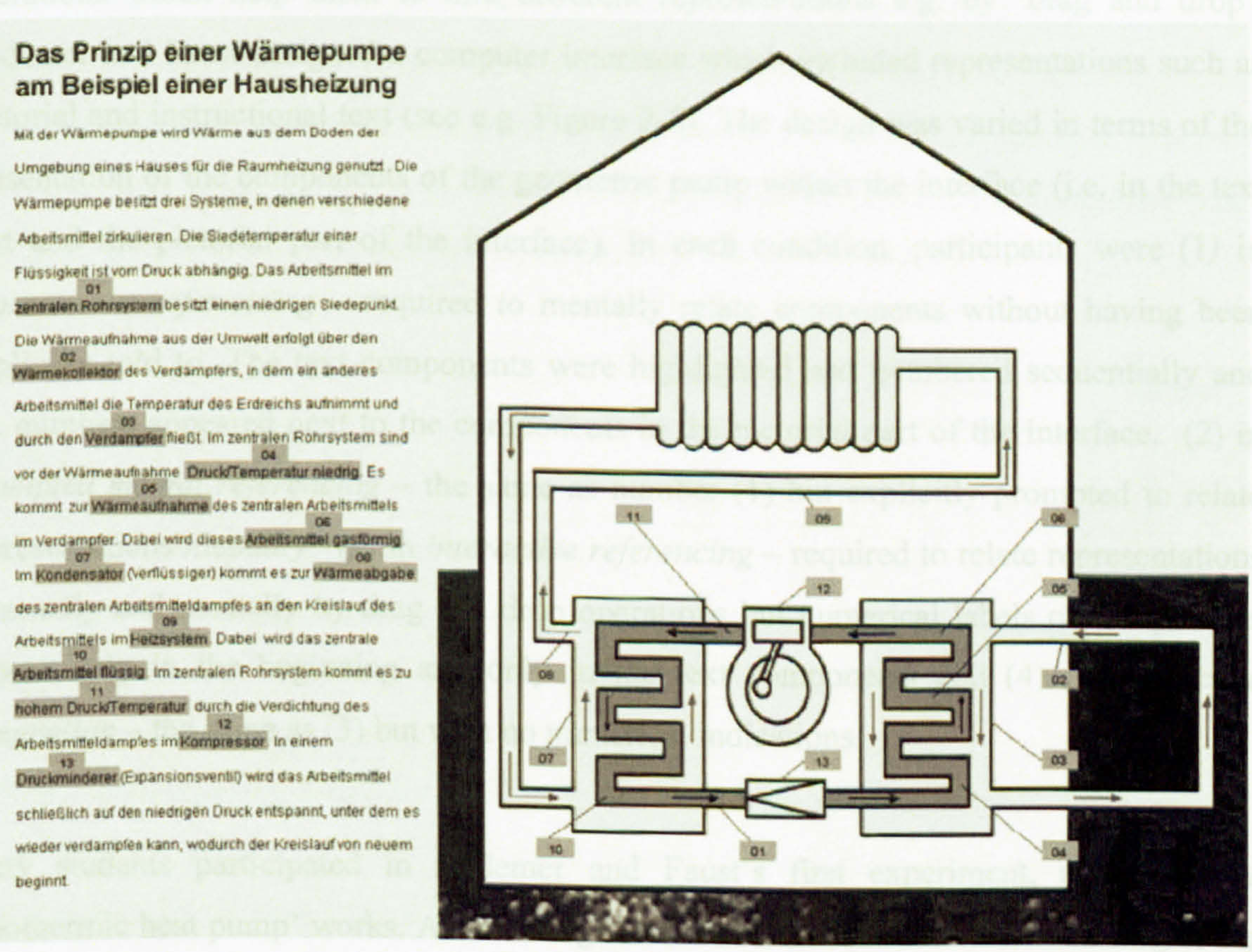


Figure 2-8 Examples of representations in the study of Bodemer and Faust (2006)

Van der Meij and de Jong's study is another example of an attempt to reduce difficulties related to visualisations of graphs. However, the value of adding interactivity to support the integration and linking of representations is not yet clear. For instance, Bodemer and Faust (2006) claim that differently presented representations can demand a considerable part of a learner's working memory; and a form of integrating representations may reduce difficulties with cognitive load. Bodemer and Faust (2006) investigated the impact of integration of representations and external activity (i.e. actions in the external environment) in two experimental studies. Bodemer and Faust were interested in the effects of adding interactivity in relating different kinds of representations. Learners integrate representations (similar to the definition given above by Van der Meij and de Jong) mentally or externally. Making representations computer based and adding interactivity features may help support integration of representations. This design could be operations which help them to link different representations e.g. by 'drag and drop'. Bodemer and Faust designed a computer interface which included representations such as pictorial and instructional text (see e.g. Figure 2-8). The design was varied in terms of the presentation of the components of the geometric pump within the interface (i.e. in the text part and the pictorial part of the interface). In each condition, participants were (1) in *spontaneous referencing* – required to mentally relate components without having been explicitly told to. The text components were highlighted and numbered sequentially and the numbers appeared next to the components in the pictorial part of the interface. (2) in *prompted mental referencing* – the same as number (1) but explicitly prompted to relate representations mentally. (3) in *interactive referencing* – required to relate representations externally and mentally by drag and drop operations but numerical labels of components appear only in the beginning and only in the text components and (4) in *interactive integration* – the same as (3) but with no numerical indications.

Sixty students participated in Bodemer and Faust's first experiment, about how a 'geothermic heat pump' works. A test was given to measure learners' understanding of the subject before and after using the software. The items in the test consisted of statements. Participants were asked whether each statement was correct or not. On the basis of the tests scores in this experiment, Bodemer and Faust found that those who relate representations only mentally have a better understanding than those who related representations mentally with 'interactivity', and those externally with interactivity. Bodemer and Faust (2006) suggest that learners are more likely to relate representations to another representation when they are familiar with the visualization required for the representations given. Bodemer and Faust also analysed talk. This analysis revealed that some of the participants in the interactive conditions were not able to relate the numbers or text components accurately to the components of the picture and thus were never provided with a correctly labelled illustration of the heat pump.

The researchers then conducted a second experiment, this time, about 'uniform and accelerated motion'. They only considered conditions (2), (3) and (4) above. This time they tested the learners for their prior knowledge. A test about uniform accelerated motion was given. Statistical tests in the mean scores between conditions 2, 3 and 4 showed that those in the interactive integration group performed better than those in the other two groups (Figure 2-9). This study shows that there are different effects on cognitive processing when representations vary in interactivity. However, Bodemer and Faust's suggest that tests may not be enough to examine the effects of varying instantiations because cognitive load during learning may need careful investigation.

		Prompted referencing	Interactive referencing	Interactive integration	Overall
Overall	M	52.6	50.5	74.2	59.1
	SD	17.8	19.2	14.6	20.0

Figure 2-9 Overall mean scores between three conditions in the study of Bodemer and Faust (2006, p. 37)

Learners may be having difficulty visualising graphs due to 'motion change'. Learners find the concept of 'motion change' difficult to grasp because they often make inaccurate mental models of moving images (Ardac and Akaygun, 2005) and integration of animation in designing computer-based representations may help in conveying change of external representation in time. It is hypothesised that animation is better than static representations (e.g. Park and Gittelman, 1992) but this hypothesis is challenged by Morrison and Tversky (2001). One-hundred and forty-four university students participated in the study of

Morrison and Tversky. The participants were divided into three groups and each group was given an instantiation in learning 'seven rules of movement'. The three display conditions were: text only, text with static graphics, and text with animated graphics. Morrison and Tversky conducted three experiments: experiment 1) text with the causal rationale for the rules of movement with movement explained as social rules or as navigation rules; experiment 2) the rationale was omitted and only the rules were described; and experiment 3) the same materials as experiment 2 but limited participants' study time. They used a 'standardised' spatial ability test (the Vandenberg mental rotation test) before and after the experiments and used the results to compare students with low spatial and high spatial ability on performance in a problem solving test about applications of the rules of movement. In experiment 1, based on the scores in the problem solving test, the group given text with animated graphics did not perform better than the text group with static graphics. In the second experiment, where participants needed to develop their understanding of the rules of movement solely from descriptions, the same result was obtained as experiment 1. The scores between text with static and text with animated graphics were not significantly different. However, in the third experiment, where time was limited, the scores revealed that text with either static or animated graphics performed better than those with text alone. The results suggest that addition of a graphic representation is better than text alone but the addition of animation as a feature did not add any value to learning. This was because there was no significant difference found in the test scores between static graphics and animated graphics in the third experiment. Morrison and Tversky suggest that dynamic instantiation, in the case of animating objects, can sometimes be ineffective.

The studies above make assumptions that there can be possibilities of addressing or reducing some of the difficulties associated with multiple representations by varying the kinds of instantiations. The studies show that different instantiations influence learning and that the effect can be due to the design and presentation of external representation, particularly animation and interactive features. There are many studies which show positive effects of varying instantiations in tackling standard external representations. However, as has been seen in the discussion of automatically generated graphs above, there are also some so far unresolved questions about effectiveness. The two sub-sections above show how changing numbers and forms of external representations may influence learning with them but the generality of findings are inconclusive. Also, many of the studies include different kinds of instantiations such as animation and manipulable objects but none of the studies identified in this section investigated animation and interactivity of multiple representations in a single study. Sections 2.3.1 and 2.3.2 above show how design of learning environments may carry with them some attempts at reducing learning difficulties with multiple representations by varying external representations and by varying instantiations. However, there may be other factors that need to be examined when learners tackle multiple computer-based representations. The next section explains that there are other factors that can be considered in attempting to reduce difficulties with multiple computer-based representations.

2.3.3 LEARNERS' SELF-CONSTRUCTED REPRESENTATIONS

Section 2.3.1 and 2.3.2 above show how the designs of some learning environments differ in terms of the combinations of external representations and instantiations. It also shows that some difficulties can be addressed by using alternative instantiations. The studies here show that learners construct their own representations even if they are offered representations.

One study investigates one learner dealing with multiple external mathematical representations in a computer environment. Borba and Confrey (1996) report a case study of a 16-year old male student working on 'transformation of functions' using software called Function Probe™. Typically, the didactic approach to understanding function, according to this study, involves an interpretation of an equation with its original graph, and the change in this original graph by modifying the numbers in the equation. For example, an original equation $y = x^2 + 6$ is modified to $y = 3x^2 + 6$, learners are asked to make a comparison between the graphs of the original and the modified equation; and then they are asked to establish a rule about transformation of functions. Borba and Confrey took a different approach by starting with graphs as the initial point of observation to see whether learners are able to make the same understanding of functions. A learner was interviewed and video recorded over a five-week period for eight times (average of twohour each session). The learner could graph functions in different ways, such as inputting equations, plotting points, sketching using the mouse, using a calculator function, and tabulating numbers. Borba and Confrey presented situations in which the learner was visualising using the tools offered by the software and situations in which the learner visualised on paper. Borba and Confrey claimed that multiple mathematical representations in a computer environment helped the learner to come up with his own way of tackling a problem. For example, the software was designed to tackle only a certain notational form of equation. The learner experienced a problem dealing with this form offered and worked with an alternative notational format. However, this approach was not supported by the design of the software. The learner opted to work on paper and attempted to check his solution. Borba and Confrey suggest that learners may come with strategies not supported by the design of the computer environment; these may be effective strategies but learners need to successfully link their own representations to those generated by the computer to generate a better understanding of mathematics. This suggests that examining the relationship between learners' attempts to tackle representations offered on the screen, and self-generated representations may provide richer information than just considering only the former. This study also showed that by using an alternative instantiation in helping a student to learn mathematics (i.e. in this study, functions) in a different may not always be enough. It was seen that the student preferred to use an approach not supported by the computer.

When learners experience difficulties with representations, they sometimes construct their own representations to try to cope with these difficulties (e.g. Mayer, 2003; Schwartz, 1995; Van Meter, 2001). Cox (1996) looked at the relationship between problem solving performance and the type of external representations that learners use. He suggests that learners' construction of their own representations could provide an indication of situations in which the given representations have been found insufficient to complete the task, or of moments at which working memory is overloaded. He also argues (Cox, 1999) that there are differences between learners' reasoning with self-constructed representations and reasoning with given representations. He emphasises the need to distinguish situations in which learners interpret representations given to them and situations in which learners construct their own representations.

According to Cox (1999),

> *"External representations (ERs) can be produced for exclusively private use or for communication with others. Learners sometimes produce and use ERs purely privately (e.g. 'workscratchings' produced on scrap paper whilst problem solving under exam conditions). Mostly, however, a person's ERs are also seen by others, and may be commented upon, as in classroom settings. ER use often represents socially shared cognition—the social embedding of the representation in an interchange between participants. A reasoner's belief about whom a representation is intended for (private versus an 'audience') will affect his or her choice of representation. Private representations are, amongst other things, less fully labelled, sparser and may be only partially externalised—whereas those intended for sharing with others will tend to be more richly labelled, better formed and more conventional." (p. 347)*

Cox outlines several factors why there are differences in reasoning with ERs between presented and self-constructed representations. Effective reasoning with ERs is a three-way interaction between:

i. The cognitive and semantic properties of the representation.

ii. The match between the demands of the task and the type of information read-off afforded by the representation.

iii. The effects of within-subject factors (e.g. prior knowledge, cognitive style).

Why do learners construct their own representations? Cox lists a number of purposes that perhaps the given representations are not fulfilling:

- reducing working memory load
- focusing attention to aspects that are particularly important to the specific task
- aiding information handling

- facilitating the inference of motion from moving displays
- facilitating shift of reasoning mode
- expressing inferences difficult to express verbally
- re-ordering information within a representation
- identifying missing information
- representing implicit information explicitly.

While Cox acknowledges the value of utterances learners make during task completion, in understanding learners' interaction with multiple representations, he also suggests that utterances may not be enough because they delimit some concepts that learners cannot express verbally. Learners construct different representations for similar problems, for example by externalising them on paper, and learners' interaction and operation with these representations differs from each other. He notes that externalisation of representations can take similar or multi-representational form. For example, representation presented in linguistic form (e.g. natural language, notation) can be 're-represented' (or recreated by learners) in the same form or can be graphical (e.g. set diagrams, tables, maps, plans, graphs) or a combination of both.

The suggestions of Borba, Confrey and Cox above highlight the importance of examining not only learners' interaction with standard external representations but also the representations they create. These latter representations provide information that can be vital in understanding which representation helps reduce learning difficulties.

It is clear that varying instantiations through alpha-numeric input and manipulability can influence learning with external representations. But studies have not been successful in determining which can be attributed to successful attempts as some of the studies have been found to have inconsistent conclusions about which particular combinations of representations and which kinds of instantiation help or hinder learning. It is not clear whether the success is attributable to the external representations or to the instantiation.

This section has shown that varying computer-based representations can be a way to address difficulties learners experience when tackling standard external representations; however, it is clear in section 2.2.2 that computerising representations can also introduce difficulties. It is important to analyse learning with these representations by distinguishing which attempts address which difficulties. When representations are combined and presented altogether, this can be complex to study. Since strategies are actually 'attempts to tackle problems', the next section makes a case that using strategy as a unit to analyse learning with varying computer-based representations can be a viable alternative to explicate the relationship between attempts and difficulties with varying standard external representations or with varying instantiations.

2.4 EXAMINING LEARNERS' STRATEGIES WITH MERS

Section 2.3 shows how the design and choice of external representations and types of instantiations are used to reduce difficulties. Furthermore, the above section provides information that as learners interact with this design, learners also construct their own representations which could provide some indications of when difficulty is experienced by them. In this section, some studies are examined in terms of the influences of varying external representations and types of instantiations on strategy choice and use, set in the context of learning. It also illustrates how strategies may be influenced by the different combinations of standard external representations and the kinds of computer operations through alpha-numeric input, animating and manipulating objects on the screen. This section makes a case that using strategies as a unit of analysis can be an effective way to understand learners' experiences with computer-based representations.

2.4.1 EXAMPLES OF INFLUENCES OF VARYING REPRESENTATIONS AND INSTANTIATIONS ON STRATEGY USE, AND EXAMPLES OF CHARACTERISATIONS OF STRATEGIES

This section shows some of the ways others have characterised strategies relating to standard external representations. The kinds of strategies elicited by the nature of external mathematical representations are given, and the kinds of findings previous studies came up with by characterising these strategies are reported.

The first study is introduced in this section to show that learners' choice of standard external representations may vary when asked to solve problems about linear functions. Keller and Hirsch (1998) administered a pre- and post-test that required seventy-nine learners to choose an external representation in approaching mathematical problems. Learners were not asked to solve the problem. They wanted to find out how learners would attempt to solve a given mathematical problem choosing from three types of external representations (i.e. graphs, numbers, and equations). The problems were paper-based, thus, instantiation is constant and static. Learners are grouped according to users and non-users of graphing technology (i.e. those learners using graphing calculator in their course and those who are not allowed to use graphing calculators). Keller and Hirsch conclude that those using a graphing technology preferred graphical representations for some problems compared with the other group. This study is based on learners' perception of the representation they would attempt to use given a certain problem. Participants were not actually asked to solve the problems. It can be argued that the representation they chose to solve a problem may not be the actual representation they choose when they actually solve a problem. They might have given a post-hoc choice of their preferred representation which they deemed could solve problems asked of them. Nevertheless, Keller and Hirsch's study seem to suggest that the intervention of technology may influence learners' choice for representations. This needs further investigation as to how learners arrive at their choice of representation when solving problems.

The next study looked at how learners construct symbolisation from graphic representations. This study characterised strategies in terms of the intervention of a computing technology – a graphing calculator. Ruthven (1990) administered a questionnaire to two groups of learners (age 11 to 19 years old; n = 80). Ruthven asked the learners to find the solutions for different mathematical graphs; and then they were asked to provide explanations for their answers. In examining learners' solutions and explanations, Ruthven discerned three different approaches in constructing algebraic symbolisations from graphs with the aid of graphical calculators. These are: 1) the analytic construction approach, where students construct symbolisation from the information available from the given graph; 2) the graphic-trial approach, where students use a graphical calculator to modify symbolic expressions by comparing successive trial graphs with the given graph; and, 3) the numeric-trial approach, where students formulate the symbolic expression from the numeric patterns of coordinates of the given graph. These approaches have clearly shown that students are approaching a problem in a procedural analytic way. The vital point to note in this study is on strategy 2). This strategy can only be elicited because of the instantiation that the technology can offer to learners. Without the possibility to modify graphs easily by changing a symbolic expression through alpha-numeric input, it may not be possible for learners to compare successive graphs, thus not possible to use this particular strategy.

Even conducted a study in (1998). This study involved 152 college students who were asked to answer open-ended questions. Out of the 152 students, ten were selected to be interviewed in depth. Even asked the learners to explain the number of 'solutions' to a mathematical problem (i.e. explaining the number of solutions of a certain quadratic function). After analysing their responses, two kinds of strategies were identified that relate to the use of graphs. These are referred to as a "Point-wise" and a "Global" approach to functions. A Point-wise approach is a strategy of learners focusing on specific elements found in the graph whilst a Global approach is a strategy on interpreting and describing properties of a graph. Even re-analysed learners' solutions from the ten interviews. She reported that many students deal with functions point-wise, but those learners using a Global strategy have a better understanding in relating graphs and equations, than those using a Point-wise strategy. To illustrate this, an excerpt taken from this study is given below where the students were asked to graph the following function:

$$g(x) = \begin{cases} x, \textit{if } x \textit{ is a rational number} \\ 0, \textit{if } x \textit{ is an irrational number} \end{cases}$$

> *Student A: (used a point-wise approach. She sketched the graph point by point) 1 is going to be 1, π is going to be 0, e is going to 0, 2 is going to be 2*
>
> *(Then she added)*
>
> *Student A: The curve is going up*

In the excerpt above, the data showed how this student relates the graph to the behaviour of the points. This kind of understanding is different to Student B who used a global approach.

> *Student B: (used a Global approach. He conceived irrational numbers as a discrete set, then he described the graph) A straight line, a diagonal line with holes in it... the rational number block should go diagonally, horizontally a series of dots.*

As illustrated above, student B described the graphs in relation to the set of rational and irrational numbers. This was an example of a student relating graphs to equations. This particular study illustrates how the identified strategies were used to explain how the use of one strategy can have a different influence in the learners' understanding of mathematical representations than another strategy.

These three studies provide ways of characterising learners' attempts to solve mathematical problems. While Keller and Hirsch's characterisation of strategies is derived according to the type of standard external representations, Even characterised two types of strategies with just one particular type of standard external representation while Ruthven characterised three types of strategies when learners link graphs with another standard external representation (i.e. algebraic representations). This shows how strategies can be characterised according to different granularities particular to standard external representations.

Two studies looked at strategy usage in greater detail than the three studies above. One of the ways in which learners attempt to solve problems with standard mathematical representations can be categorised is into either an "algebraic approach", in which learners' preference geared towards algebraic symbolisation, or a "visual approach", in which the learners prefer to focus on graphical representations (Villarreal, 2000). An investigation of three pairs of learners' task completions about 'Differentiation of Functions' in a computer environment (Derive™) is given here as an illustration of how one can analyse transcripts based on video-records. The characterisation of strategies were mainly identified in terms of whether learners are using either algebraic notations – algebraic approach in solving a problem; or using graphs – visual approach. Villarreal provided some evidence that learners' strategies may change from choosing algebraic or visual strategy when tackling mathematical representations. In the analysis of transcripts and videos, Villarreal noticed that when learners are using an algebraic approach, use of computers can lead to discomfort in manipulating algebraic symbolisations and result in a shift to a visual approach. Villarreal suggests that strategies can be used to find out what made learners shift from an algebraic approach to a visual approach in terms of the presence of technology. She claims that technology can influence learners' choice for using representations and believes that learners who prefer algebraic approaches could experience discomfort in a technological environment. This can be further examined to unpick the differences in the kinds of instantiations that may influence the change in strategy choices.

To present another characterisation of strategies, this study explores the issues of strategies in investigating how learners solve mathematical problems (Senk and Thompson, 2006). Written solutions of 306 secondary students were analysed in terms of the strategies they used. The test items were about four different problems on 'Non-linear Functions'. According to Senk and Thompson, students use strategies differently from each other between problems. In certain problems, some students were using strategies tackling algebraic representations whilst some were tackling numeric or graphic strategies. The results of their study indicate that learners try out different strategies and that learners can decide the viable strategy for specific problems. The researchers categorised the type of strategies into Algebraic, Numeric and Graphic. They related the strategies that students used to their test performance. Senk and Thompson acknowledge the need for studies in explaining how learners' strategies are influenced by the format of learning materials given to learners.

Villarreal points out that learners can change strategies within a task while Senk and Thompson say that learners try our different strategies before deciding which of the strategies to pursue. However, there are also some claims that learners using multiple strategies are better than using a single strategy (Tabachneck, Koedinger and Nathan, 1994). Tabachneck and her colleagues asked 12 university undergraduates to solve mathematical problems on linear function (e.g. 'money problem': a man has three times as many quarters as he has dimes etc.). They identified four different strategies namely algebra, guess and test, verbal math and diagram using verbal protocols. These strategies are based on the external representations given to their participants. Tabachneck et al. propose a **'multiple strategy effect'**; students using more than one strategy were more effective in solving the problem given. They suggest that multiple strategy effect is only applicable for non-routine or complex tasks. Learners use strategies that convert verbal representations into an alternative form either in algebraic, or verbal or visual (i.e. diagrams) (Tabachneck, Koedinger and Nathan, 1994).

This section, so far, provides a case that strategies can be categorised into different standard external representations (e.g. Keller and Hirsch, 1998) but also point out that within each standard type, strategies can be further categorised at a finer level (Even, 1998). While there are studies that show learners can choose one strategy to solve a problem, there have been some claims that learners can combine different strategies (Ruthven, 1990; Tabachneck et al., 1994) which may be more advantageous than using only one strategy. This section thus supports the examination of strategies with different standard external representations. However, there are factors about the type of technology and instantiations which could be causes of change in learners' choice of strategies.

Novick (1990) investigated how learners relate new problems to specific example problems that they previously encountered using representations. She mentioned that typically the application of learners' acquired strategy to a new problem is related to a specific solution to a familiar problem. That is, if two problems are similar, the strategy used in one problem may be carried over to the next problem. She distinguished three types of solution aids:

i) general solutions strategy – e.g. a strategy chosen by working out how an answer was derived

ii) solution procedures – e.g. application of a mathematical formula or sketching a graph

iii) problem representations – e.g. particular representation adopted placing constraints on a solution procedure.

She wanted to find out whether learners can also transfer strategies with graphical representations from a previous problem to new 'non-similar' problems. Thirty learners participated. Three non-similar problems were given. One group was exposed to the use of 'matrix' (a type of graphical representation) which could be used in their strategy to solve the problem while the other group was not exposed. The target problem (the problem which was analysed whether transfer of representations occurred) was about 'probability'. Novick found that those who received the matrix were much more likely to use the representation than those who did not. The study suggests that learners can apply a strategy with representation and that a particular strategy can be applied to non-similar problems. The categorisation of strategies provides a helpful distinction. However, the categorisation between solution procedures (strategy category 2) and problem representations (strategy category 3) do not seem to be too distinct as the latter categorisation impacts on the former. In the examination of strategies with multiple external representations, Novick's categorisation can be further extended in the light of whether instantiations can influence strategy choice with different representations. The next study presents a case with which instantiations influences strategy choice.

In an experimental study by Lewalter (2003) with 60 students, she investigated the effects of static and dynamic visuals in an expository text. The topic is related to 'astrophysics' and the focus of the analysis was on students' strategies and thought processes. Two groups were exposed to text followed by static visual for the first group and dynamic visual for the second group. The dynamic visual consisted of animated phenomenon about astrophysics whereas the static visuals are a series of 'frame-by-frame' pictures of the animated ones. In the analysis of talk, she found no indication of significant difference on learning outcomes between dynamic and static visuals. The results show that both instantiations are equally effective in supporting the learning process. However, the use and choice of strategies varied between the groups. The kinds of strategies examined in this study were not about external representations but about metacognitive strategies. One example of metacognitive strategies she characterised was 'rehearsal strategy' when an utterance is a repetition of a previous 'think- aloud' data (a kind of verbal data which will be further explained in chapter 4). With just this particular strategy, Lewalter suggests that learners use varied between instantiations. However, Lewalter suggests that further work needs to be done on the influences of dynamic instantiations in relation to graphical representations because the topic 'astrophysics' instantiates text into visuals of a graphical nature. Whether graphical representations, as static or animated, make a difference in strategy choice remains a question in her study. According to Lewalter, what particular components on the visual display led students to use different strategies may also be crucial to the understanding of the concept.

None of the studies mentioned above characterised strategies with three different standard external representations into finer level of granularities and how the use of these strategies are influenced by the type of instantiation of these external representations. This section supports the possibility of using strategy to analyse learning with multiple computer-based representations.

2.4.2 OTHER CHARACTERISATIONS EXEMPLIFYING INFLUENCES ON STRATEGIES

There are other characterisations of strategies that may help understand the effect of varying computer-based representations. The studies given below may help aid the understanding of what other possible range of strategies can help explicate the relationship between standard external representations and instantiations and the difficulties related to them in terms of cognitive load, and interactivity.

One of the standard external representations being dealt with in this research is the graph. Learners use different strategies related to graphs by visual imagery. DeWindt-King and Goldin (2003) observed two children solving problems about fractions. They reported the strategies used by these two children in a longitudinal study related to 'visual imagery'. They identified the following imagery strategies and defined them based on other studies (i.e. Owens, 1993, Presmeg, 1985).

1) Concrete/pictorial imagery – learners externalise internal imagery through utterances and gestures as if it was a physical object or picture.

2) Pattern imagery – learners present aspects of a problem in a visually schematic way

3) Memory images of symbolic notation – learners visualise formal mathematical expressions, e.g. a mental picture of a formula and reading the image.

4) Dynamic imagery – learners report of active movement of a part of an image

5) Mental operations or transformations – learners modify image as a result of visualising a succession of static images.

DeWindt-King and Goldin speculates that these strategies are not mutually exclusive. Among the strategies they looked at, their results show that the children's use of the 'memory images of symbolic notation' strategy develop over time. They expected this result as children's experience with symbolic notation matures in school. The strategies identified by DeWindt-King and Goldin above illustrate the kinds of strategies that can have impact on different forms of instantiations. For example, dynamic imagery strategy (4) can be a strategy when learners visualise a moving part of a mathematical graph is different from a visualisation strategy to do with static images i.e. mental operations or transformation strategy (5 above). This supports the investigation of these kinds of strategies in this research. However, another form of visualisation not identified above is the visualisation that happens when learners visualise movement of a graph in their minds. A possible way to externalise this is by looking at strategies related to learners' gazes.

Gaze can now be captured using infra-red cameras (e.g. Tobii, 2003). The infra-red cameras track the pupil of the eyes and relatively map it on an accurate location on the computer screen. Gaze can be externalised as a video record which shows the parts of the screen looked at by a person. An eye-tracking study showed that learners' use mental imagery during problem solving (Yoon and Narayanan, 2004). Yoon and Narayanan presented two problems: the first consisted of a graphical representation (a picture of a mechanical device) and the problem described in text; whilst the second contained only the question (Figure 2-10). Amongst the 90 learners they eye-tracked, they identified two groups, the imagery group (n = 31) and the non-imagery group (n = 57). The former group used what they called an 'imagery strategy.' Imagery strategy, as categorised by Yoon and Narayanan, is when learners' eye-movements systematically scan the empty area of the screen where they previously had seen the picture of the mechanical device. Yoon and Narayanan suggest that systematically scanning relevant elements of a visual display is an indication of a good strategist. The data of Yoon and Narayanan illustrate that the 'gazes' on the blank screen of those in the imagery group lie directly on or close to the relevant elements of the display. This is similar to the data of Johansson et al., (2005), Laeng and Teodorescu (2002). Although Yoon and Narayanan did not find any difference between the two groups in terms of accuracy in answering the problem, they found that the imagery group took longer time in solving the problem than the other group that did not use an imagery strategy. Yoon and Narayanan highlight the fact that those successful problem solvers of the imagery group paid more attention to significant parts of the representation shown than the unsuccessful ones. Some of the limitations they acknowledged in their study were: the simplicity of the task given; and the possibility of the experience with the first problem being carried over to the second problem. Yoon and Narayanan believe that in problem-solving, learners can be categorised as those preferring to use an imagery strategy and those that do not. The characterisation of this strategy together with that of DeWindt-King and Goldin may provide a different form of evidence in looking at learners' choice of strategy with mathematical graphs. It could be a case that learners may use different imagery strategy when different instantiations are involved. A similar study substantiated this is presented below.

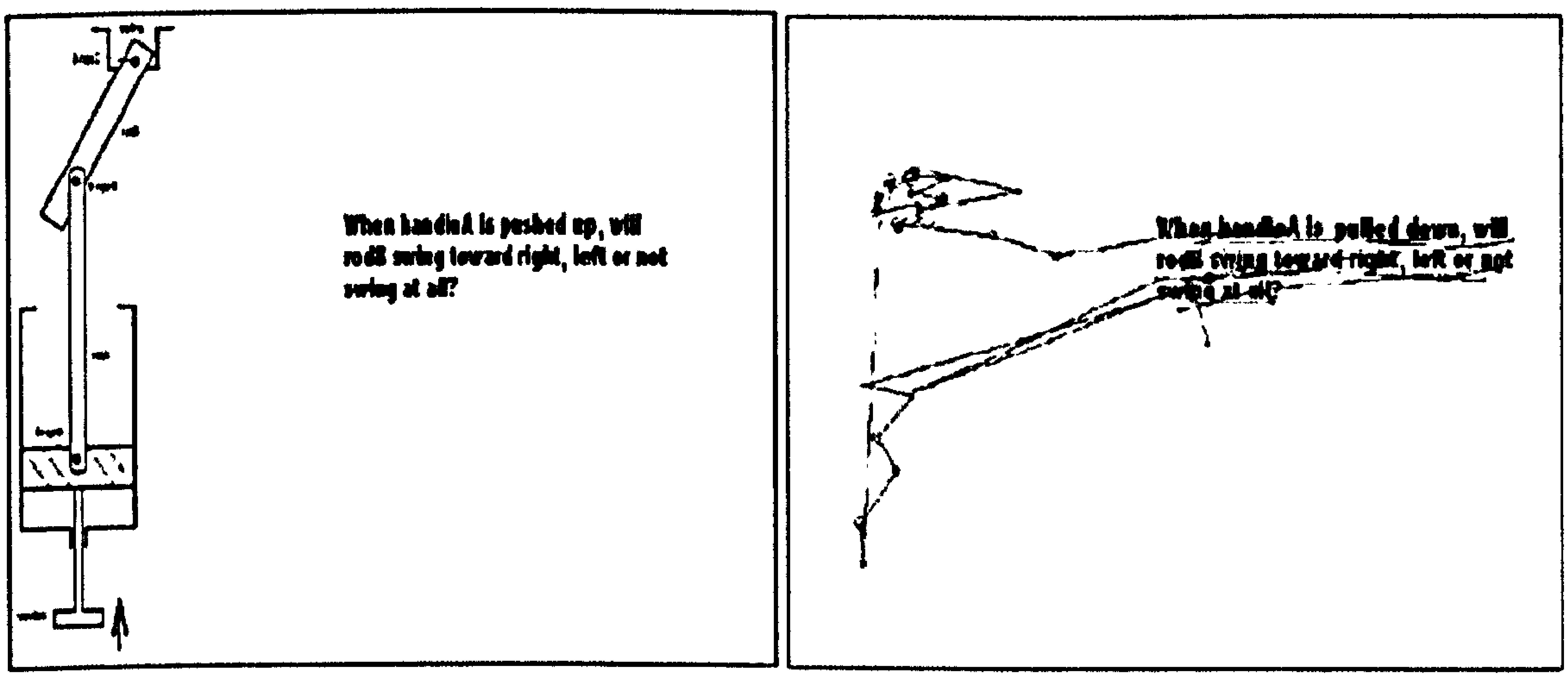


Figure 2-10 Example of a scanpath in the study of Yoon and Narayanan

According to Laeng and Teodorescu (2002), eye-movements during mental imagery assist the process of image generation. They conducted two experiments where students viewed pictorial representations, an irregular checkerboard in the first experiment, and colour pictures of tropical fish in the second experiment. They tracked participants' eye movement during and after viewing. Participants were divided into three groups: two of the groups during viewing were asked to maintain fixation onto the centre of the screen, participants in the other group were free to move their eyes. After viewing, they tracked participants' eyes and asked to recall precisely what participants saw. For those who were free to move their eyes, Laeng and Teodorescu found an association between the percentage of time spent on fixations on specific locations during viewing and time spent on the same 'empty' location during recall. Those who were asked to fixate at the centre of the screen remembered less than those who were free to move their eyes. Figure 2-11 below shows examples of scanpath during viewing (upper-left – free eye movement, lower-left central fixation) and during recall (upper-right – free eye movement, lower-right central fixation). Laeng and Teodorescu showed that eye scanpaths during visual imagery re-enact those of perception of the same visual scene. This finding is important to note. Representations such as mathematical graphs may require learners to remember certain locations of a part of a graph. When learners compare a previously seen graph with a currently displayed graph, eye scanpaths can provide some information about the cognitive processing learners are attempting in relating parts of certain graphs.

> *"Commands to the eyes for each fixation are stored along with the visual representation and are used as spatial index in a motor-based coordinate system for the proper arrangement of parts of an image." (Laeng and Teodorescu, 2002, p. 207)*

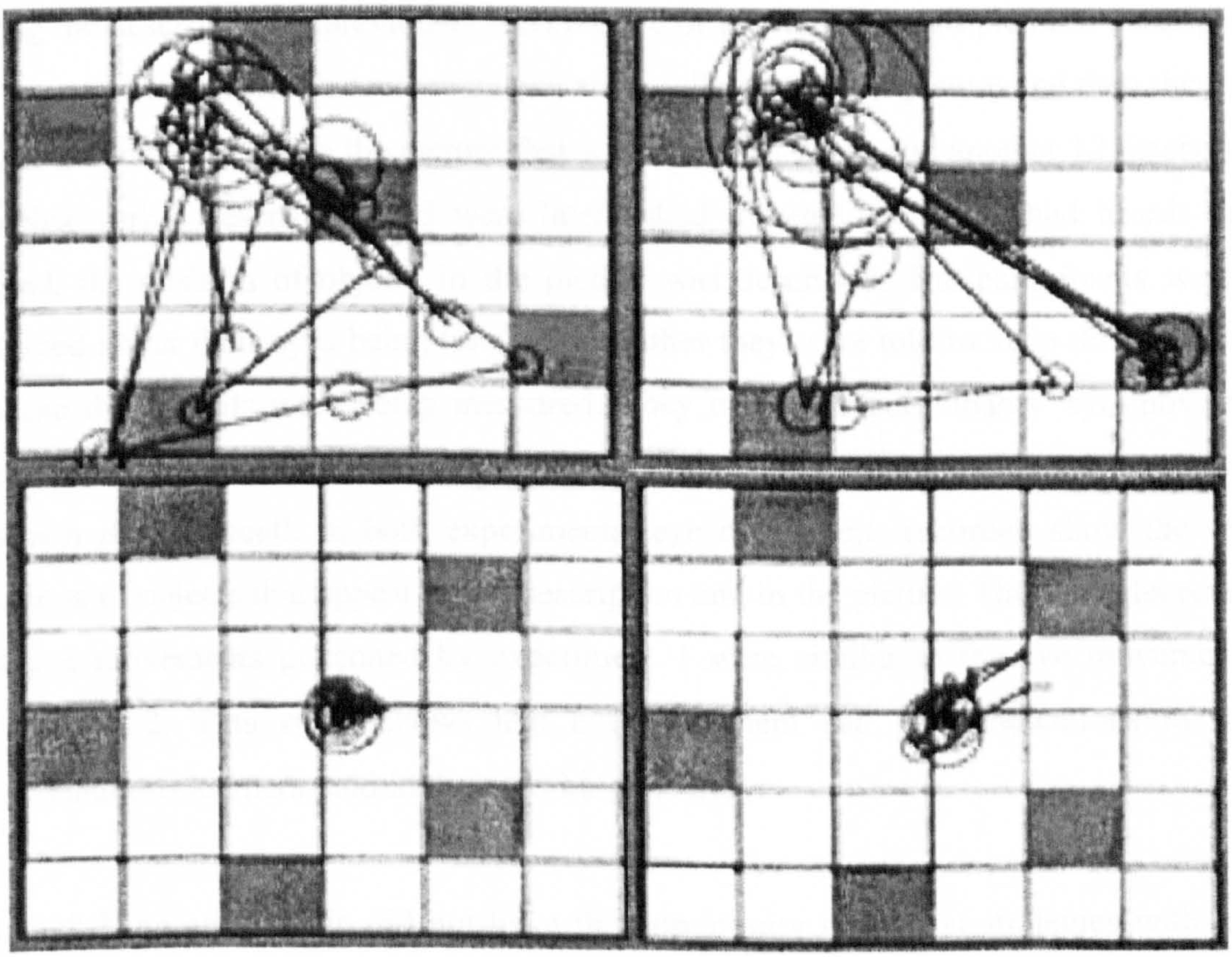


Figure 2-11 Examples of scanpaths from the study of Laeng and Teodorescu (2002)

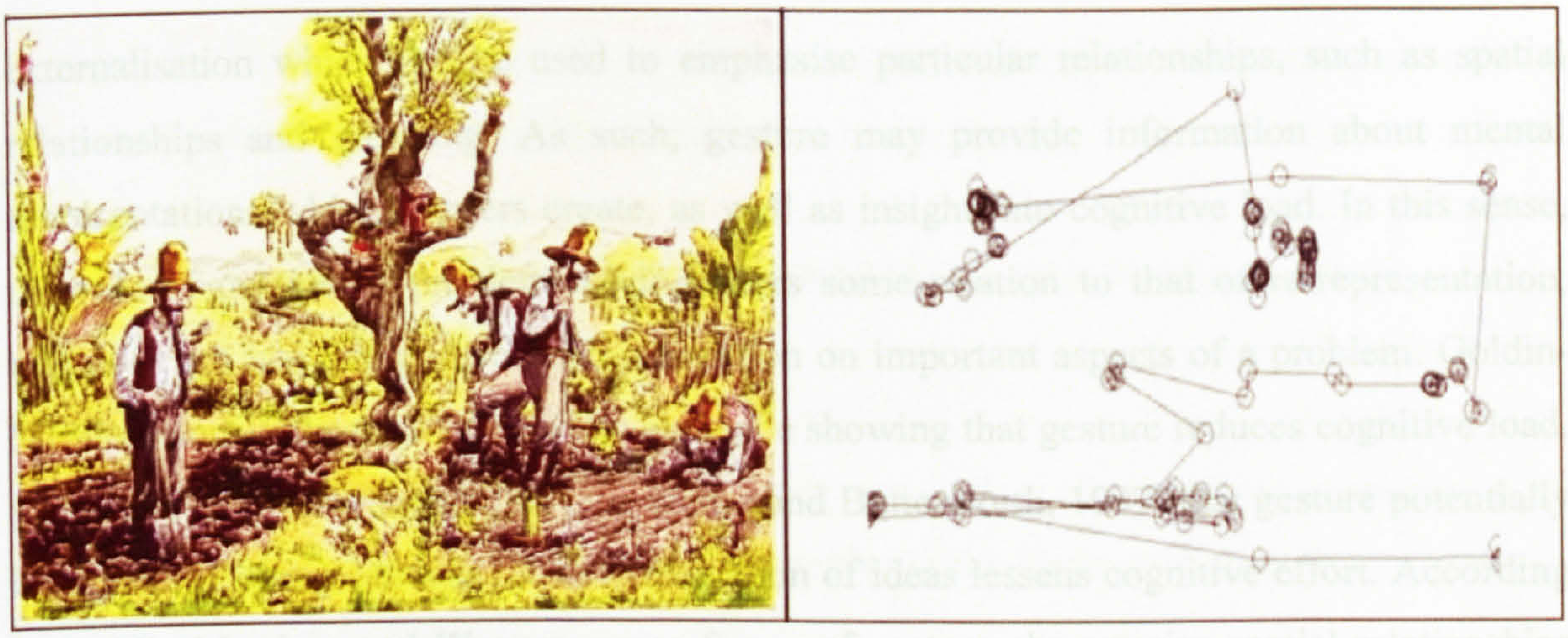


Figure 2-12 The picture stimulus and the scanpath when recalling objects previously seen or heard from the study of Johansson, Holsanova and Holmqvist (2005)

Johansson, Holsanova and Holmqvist (2005) confirm findings of Laeng and Teodorescu (2002). They both provide evidence of eye-movements reflecting the positions of objects during the description of previously seen picture or previously heard pictorial descriptions. In two experiments, 12 students were presented with a complex picture and then they were asked to verbally describe the picture that was shown to them; and another 12 listened to a complex verbal description and were later asked to retell what that had heard. In the second, the position of objects in the picture was described. The participants were not informed about their eyes being tracked but rather they were told to keep their eyes open because their pupils were being measured. They examined participants' eye-movements while recalling objects previously seen or mentioned. Figure 2-12 shows an example of scanpath during recall. In both experiments, eye movements recorded show the spatial locations of objects that appear in the description and in the picture. These results revealed that eye movements generated by experiment 1 were similar to the eye movements in experiment 2. This study shows that eye movement can give indications of visual representations learners hold in their working memory.

Dewindt-King and Goldin did not have the opportunity to look at strategies with gazes. However, they have identified a 'Concrete/pictorial imagery' strategy through utterances and gestures. Gestures, such as hand movements, are used as aids in cognitive processing (Goldin-Meadow, Nusbaum, Kelly and Wagner, 2001). Gesture is viewed as a form of externalisation which can be used to emphasise particular relationships, such as spatial relationships and grouping. As such, gesture may provide information about mental representations which learners create, as well as insight into cognitive load. In this sense, the role of gesture in problem-solving bears some relation to that of re-representation, because each has a role in focussing attention on important aspects of a problem. Goldin-Meadow and Wagner (2005) review evidence showing that gesture reduces cognitive load. They deduce from other studies (e.g. Hadar and Butterworth, 1997) that gesture potentially influences learning because the externalisation of ideas lessens cognitive effort. According to Goldin-Meadow and Wagner, some forms of gesture characterise spatial relationships particularly when a task is spatial in nature. In (2001), Goldin-Meadow, Nusbaum, Kelly and Wagner analysed gestures of twenty-six children on 'addition' problems (4 + 3 + 2 = ?) and thirty-two adults on 'factoring' problems (x^2 – 5x + 6 = ?). They asked participants to solve the problems on the blackboard. After solving each problem given, participants were asked to remember a list of words. The words were read out loud for children; a set of letters were shown on card for adults. They measured the demands on cognitive load imposed when explaining each problem whilst at the same time remembering the list of word/letters. Participants gave explanation under 1) gesture permitted and 2) gesture restricted conditions. They revealed that participants remembered more items when they gesture than when they did not. Goldin-Meadow et al. assert that gesturing helps reduce cognitive load because it appeared to save cognitive effort needed when explaining. In learning situations, learners may self-explain as they process information mentally. This highlights the importance of being able to characterise different types of imagery strategy relating to gazes and gesture as these strategies may help in processing different representations. So, gesture and eye movements can be viewed as externalisations of mental activity and hence can reveal things about cognitive load and task performance. Importantly, they can reveal where attention is paid, and where learner difficulties arise.

The studies identified in this section have established that strategies through gazes, sketches, utterances and actions can provide ways of characterising imagery strategies. However, the relationship of these strategies with standard external representations needs to be examined as learners do not use these strategies in isolation from each other. Different types of instantiations may have an impact on the kinds of imagery strategies that learners use. The inclusion of these strategies in examining the influences of varying instantiations with these strategies should not be overlooked.

2.4.3 ANALYSING LEARNERS' STRATEGIES WITH COMPUTER-BASED REPRESENTATIONS

The preceding sections suggest the relevance and value of examining strategies with multiple computer-based representations. Examples of the kinds of analyses, using strategy as a unit, are given below.

Weigand and Weller (2001) used strategy to look at learners working with computer-based mathematical representations with 11th grade students. The study focuses on the changes of learning strategies during problem solving, and in the possibilities of developing a technique based on 'computer protocols' to evidence changes in learners' strategies. The technique they used consisted of video-recording during the activity, a 'screen capture' in real-time and logging of computer operations. The activities and Computer-Algebra Systems (CAS) software used were: Derive™ – for 'quadratic function' (application of maxima and minima); and for 'applied problems' (e.g. the motion of a twine handle of a sewing machine); and Mathplus™ – for 'trigonometric function' (application of sine function). They offered the learners the freedom to decide when to use the software. The study compared static representations generated by learners using 'pencil and paper' versus computer-based representations via keyboard input and mouse operations.

By looking at learners' strategies with standard external representations and with representations learners generate on paper, and relating this to the influence of making representations computer based, they provide a rich account of learning interaction with multiple representations. For example, Weigand and Weller provide evidence that learners tend to use more representations with the computer than with pen and paper. Their investigation also shows that learners switch between different representations more when they are using computers than when they are using pen and paper. They also identify that learners often start with graphs and view graphs as a pictorial representation. Weigand and Weller claim that learners do not necessarily have a better understanding of the mathematics concept with computer-based representations; but their understanding is different from that of learners using 'pencil and paper'. Weigand and Weller also highlight the usefulness of the technique they used (i.e. video-recording, computer screen capture and computer logs) in analysing and categorising strategies. Weigand and Weller make a case that representations that learners choose are different when representations are computer-based or paper-based. In relating this study to the study that Weigand (1991) conducted earlier, Weigand and Weller manage to provide an alternative explanation as to why the ease and speed of computers to show different representations fail. According to them, the many representations generated in a small time possibly overwhelm learners and make them interpret and read representations in a less reflective way. Weigand and Weller conclude that

> *"empirical investigation showed repeatedly that learners sitting in front of a computer seldom have the patience to read representations on the screen, and then to interpret and reflect on them" (p. 109).*

Also, in terms of the findings of Van der Meij and de Jong (2006), claiming that dynamically linked representations are better than non-linked ones, examination of strategies provide additional explanation that when learners actively integrate the representations themselves, a difference is found. Weigand and Weller looked at learners' strategies with standard mathematical external representations and with representations they create on paper.

Aczel (1998) presents a case that examining strategies is one of the possible ways to explain the difference in the effect of representations in learning. Aczel's research on examining learners' strategies is a key study related to this research. 100 learners aged 10 to 15 joined this study. A typical task involved finding the numeric value of an 'unknown', an algebraic variable. Aczel developed computer-based software that trained students to learn equations using a 'balance model' (use of a 'weighing scale' metaphor as game-like balance model). The students engaged with the software by balancing the weighing scale by removing and adding 'weights' and 'barrels' at either side of the scale (see e.g. Figure 2-13 below). The weights had corresponding numerical values whereas the barrels have an 'unknown' numerical value. By engaging learners with these kinds of representation, it is intended to shift their thinking to algebraic and numerical notations as they progress through using the software at different levels of difficulties. For example, the first level requires learners to deal with weights and barrels and then further on to the upper level the weights and barrels are replaced by numbers and letters (i.e. representing algebraic expressions). It is intended to encourage learners to come up with algebraic related strategies. The learner has to type in the values for each barrel and check how the display on the screen changes. Although mathematically the representations involve equations, the program requires the learners to shift their thinking with one form of representations e.g. from that of barrels with values to an algebraic form of representation.

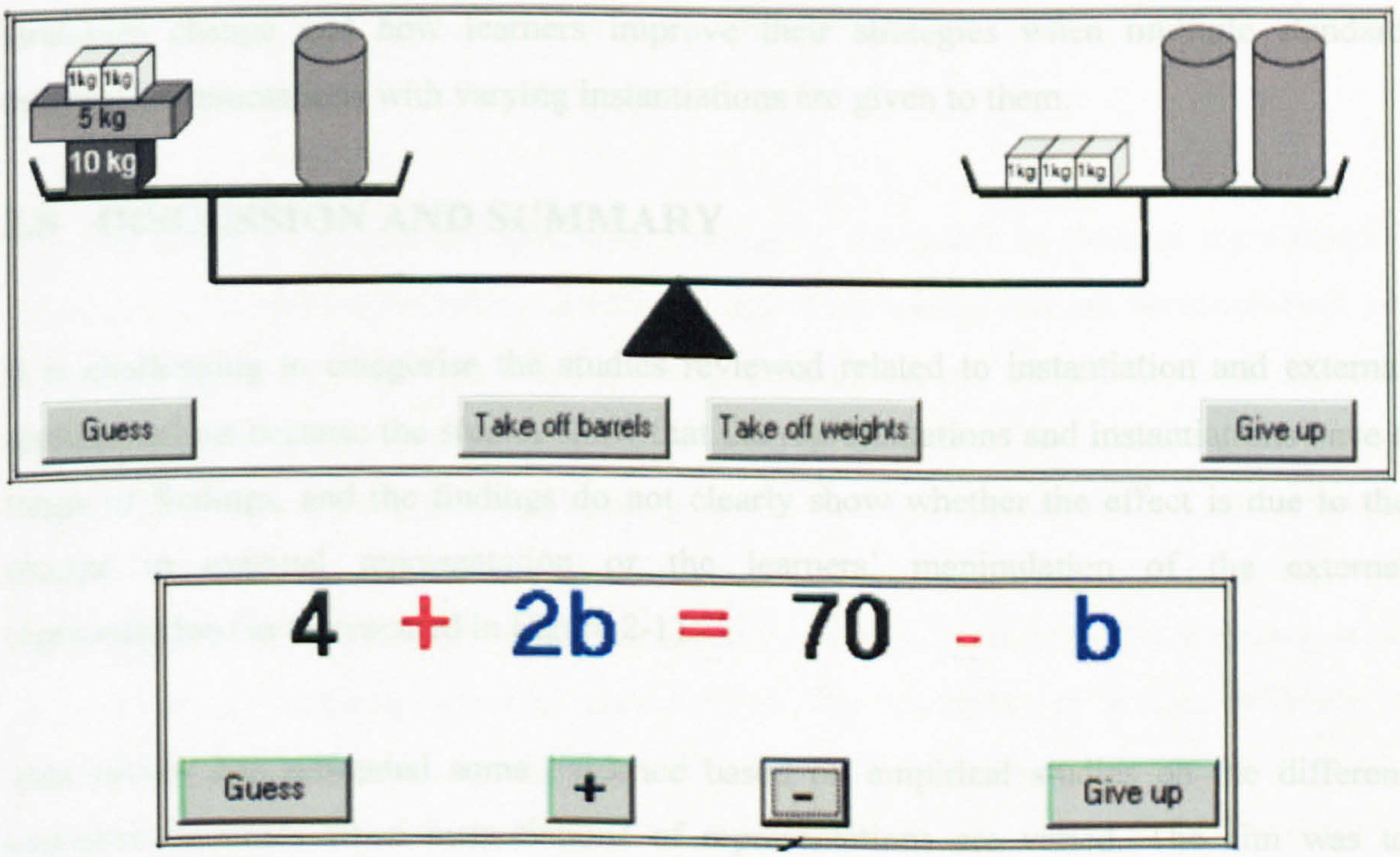


Figure 2-13 Examples of representations in the study of Aczel (1998)

One of the techniques used to gather data was observation of learners as they use the software. The software was also designed to record logs of students' actions. Based on these logs and on interview recordings, responses to written tests and questionnaires, recordings of conversations of students working at the computer, the development of algebraic strategies from using the software was analysed. For example, data from a pair of learners in the study showed that the kinds of strategies that learners choose when tackling barrels and weights to the kinds of strategies that learners choose when tackling numbers and variables were transferable. The kind of instantiation was not varied but the kinds of standard external representation changed. The same kind of manipulation may help learners to apply the same strategy even though the external representations have changed. This study illustrates how it is possible to be able to examine the influence of one particular instantiation by looking at strategies on the kinds of external representation learners are tackling. Aczel argues that learners use strategies in order to address a particular concern. This can be extended when learners are not only concerned with algebraic representations. Aczel suggests that learners subject strategies to 'trial-and improvement' in dealing with a particular concern. It would be interesting to examine how strategies change and how learners improve their strategies when multiple standard external representations with varying instantiations are given to them.

2.5 DISCUSSION AND SUMMARY

It is challenging to categorise the studies reviewed related to instantiation and external representations because the studies show that the representations and instantiations have a range of findings, and the findings do not clearly show whether the effect is due to the change in external representation or the learners' manipulation of the external representation (as represented in Figure 2-1).

This review has presented some evidence based on empirical studies on the different cognitive demands when instantiations of representations are varied. The aim was to provide enough justification for researching the effect on strategies of varying computer-based representations. It critically assesses key research relevant to how learners visualise, manipulate, and use multiple representations that are differently presented and instantiated. It identifies the gap in the literature related to the influences of varying instantiations on representations. The review demonstrates that the use of strategies with multiple representations is under researched and that strategies themselves have been very little used to study the effect on learning with computer based representations.

Key findings related to difficulties with standard external representations and to the kinds of instantiations are identified. There are studies claiming that learners can benefit from MERs. However, the literature suggests that a careless combination of design and external representation in a computer environment can take this benefit away. External mathematical representations have been made available via computer and are widely studied. Until now, educators and researchers alike are still faced with the challenge of understanding how learners process mathematical representations in a computer environment. The conflicting claims about external representation and instantiations present a challenge. There is a need to study representations and instantiation together; and measuring learning outcomes and performance may not provide enough information about the complex interaction between representations and instantiations. What is not clear from the literature is which difficulties are due to the nature of external representations or to how these representations are instantiated.

The possibility to address the difficulties relating to the nature of external representation and linking multiple representations with the aid of computers has not been achieved yet. Moreover, it seems that computers are also introducing a different form of difficulty relating to manipulation and operation depending on the type of instantiation. In the ways that difficulties can be grouped according to the standard representations, the strategies learners use with them is also distinguishable. The studies show how strategies may vary depending on the standard external representation and the characterisation of strategies can be defined into different levels of granularities. The possibility of getting evidence for describing how instantiations might be able to influence learners' strategies in solving problems needs further investigation.

There is a gap in the literature that incorporates the investigation of Static, Dynamic, and Interactive instantiations of all three commonplace external mathematical representations into one study. Moreover, not many studies were found to use strategy in researching learning effects with multiple representations. There is a lack of research involving strategies with multiple representations and on how instantiation influences learners' use of strategies. Examples of the characterisations of strategies with standard external representations in different levels of granularity are identified in this chapter. Aczel (1998) provides a guiding framework to examine strategies with multiple representations (i.e. to examine how strategies change and how learners improve their strategies when multiple standard external representations with varying instantiations are involved).

Chapter 2 has provided five different hypotheses about strategies and computer-based representations. These are:

1) Strategies with each standard external representation can be characterised at different levels of granularity

2) Learners' choice of strategies depends not just on the standard external representations given but also on the instantiation

3) Mental constructions of images with graphical representations vary between instantiations.

4) Attention paid with each standard external representation varies between instantiations.

5) Expression of inferences varies depending on the instantiation

San Diego (2003) (discussed in section 1.4) investigated the kinds of strategies learners use with different graphing technologies. However, methodological insights are suggested by this study. It is hypothesised that analyses of strategies based on gazes, actions, utterances and sketches can identify factors contributing to strategy choice in a way that is not possible with traditional observation techniques

The external representations involved in this research are multiple external representations and it is vital to tell exactly which representation is being tackled. In the next chapters 3, 4 and 5, this methodological challenge of exactly telling which representations and which instantiations influence strategies when learning with mathematics are dealt with in detail.

This research aims to provide evidence based on multiple sources of digital 'data' (i.e. gazes, utterances, actions and sketches). The next chapters examine how the kinds of strategies related to gazes, actions and sketches may help provide evidence. It is important to take on the suggestions of some of the studies in this chapter to research strategies related to imagery, interactivity, re-representation and external representations.

3 INNOVATIVE APPROACHES TO CAPTURING LEARNERS' COMPUTER INTERACTIONS

3.1 INTRODUCTION

In some learner-computer interaction situations, learners look at the screen, write, type, use the mouse, and may even talk and show gestures of joy and frustration. When multiple computer-based representations are involved, to be able to understand the dynamic of such situations, one has to capture what the learners say, do, see, and write. However, it can often be extremely difficult for a researcher to observe and systematically code multiple forms of simultaneous research data.

The approaches to data capture developed in this research attempt to minimise such problems by means of recent technologies. This chapter considers how digital approaches to capturing video data of learners' computer interactions can offer some benefits. It illustrates how a Tablet PC can enable real-time writing and sketching to be captured. It also illustrates how the latest technologies facilitate eye-movement recordings to identify objects of attention (Hansen, Hauland and Andersen, 2001) without the need for cumbersome head-mounted devices. By capturing what learners say, do, see and write, the approaches to data capture may help identify the subtle differences of the strategies learners use when tackling computer-based mathematical representations.

The chapter begins by outlining the reasons why video is used to research learners' computer interactions, and the typical challenges faced. Digital approaches to capturing video data are then described in detail.

3.2 THE USE OF VIDEO TO RESEARCH LEARNERS' COMPUTER INTERACTIONS

This study identified use of video to research learners' interaction because videos can capture simultaneous forms of data. However, it is important to be aware of the advantages and the challenges of using videos to research learners' computer interaction.

3.2.1 ADVANTAGES OF COLLECTING VIDEO DATA IN RESEARCH

There are numerous reasons why researchers collect video data. In particular, it has been regarded as an effective way of recording people's interaction with technology (Neal, 1989). Though video may not provide a complete record of behaviour because of its restricted visual range, it offers for some researchers the potential of a more complete and consistent record than observation notes (Foster, 1996). Unless the phenomena under scrutiny are particularly slow-moving, the quality of notes depends crucially on the note-taker's instantaneous ability to detect the significant phenomena as they happen, decide what are the key aspects that should be recorded, and record them quickly, without missing further events, and all perhaps while conducting the fieldwork and identifying follow-up questions.

Video can also capture events that would otherwise be difficult to record (Mackay, 1988) such as gestures, speech patterns, and glances. In addition, multiple cameras allow the capture of simultaneous behaviour or multiple perspectives that would be impossible for one researcher to see (Powell, Francisco and Maher, 2003). Moreover, video helps ensure some consistency of recording, compared with the use of multiple observers.

An important advantage of video over observation notes is that it allows the researcher to delay until analysis some of the selectivity and interpretation that would otherwise be concurrent with the phenomena under observation (Powell et al., 2003). Moreover, the choice of video means that data can be repeatedly replayed (Jacobs, Kawanaka and Stigler, 1999; Plowman, 1999; Roschelle, 2000), at leisure, and at different speeds. Video data can be time-sliced (Jacobs et al., 1999), used to generate specific quantifiable behavioural data (McLarty and Gibson, 2000), and micro-analysed frame-by-frame (Ratcliff, 2003).

Video and observation notes can both be subjected to inter-rating; that is, different people coding the same behavioural data, to check codings and to enable multiple perspectives to be brought to bear on the data (Powell et al., 2003). However, just coding observation notes without seeing the actual event is arguably less accurate than coding notes with a video of what is being coded.

Other uses of video data include triangulation with other sources (Plowman, 1999), validation of interpretations (Foster, 1996), and a stimulus for discussion (McLarty and Gibson, 2000).

3.2.2 CHALLENGES IN TRADITIONAL APPROACHES TO USING VIDEO IN RESEARCH

Although there are numerous reasons why researchers are using video data, there are also disadvantages. There are technical, methodological, practical and ethical challenges that arise from the use of video in research; and also some technical and practical challenges depending on whether the type of video format is either analog or digital video.

Technical challenges: Technical challenges to the use of video have been a disincentive for some researchers. Selecting, setting up, and operating video equipment demands new skills (Bigum and Gilding, 1985; Knoll and Stigler, 1999; Mackay, 1989; Roschelle, 2000; Spiers, 2004). When the format is analog, without careful adjustments, there can be mismatches between monitor refresh rates and camera frequencies, resulting in flickering video playback of a recorded computer activity. In digital format, there can be mismatches between video codecs. This means that a video stream encoded with a specific codec can only be watched when the exact codec is present and properly installed in the machine; otherwise, the video playback is not possible.

Practical challenges: Transporting, positioning, testing and looking after equipment can take time and effort. The data storage media need to be managed appropriately, with the need for a good labelling and indexing system, backups, storage space, care to avoid degradation in quality, and the like. Analysis of video data also presents challenges. Analogue recordings on VHS tape, for example, can be difficult to search and compare. Hardware to allow accurate indexing, frequent pauses and rewinds, and the simultaneous display of multiple images can be expensive. Transcription and coding can be time-consuming (Knoll and Stigler, 1999; Plowman, 1999); some have estimated that this can take 10 times the length of the video (Roschelle, 2000). Finally, presenting and reporting results from video analysis in written form can be challenging (Bigum and Gilding, 1985; Mackay, 1989; Powell et al., 2003). In digital recording, when a person wants to convert one video file to another format, video conversion may also require time processing.

Methodological challenges: It is well-known that people can behave differently when they know they are being observed, perhaps even more so when they are on camera (Bowman, 1994; Foster, 1996). The size and positioning of video equipment, and changes needed to light and sound conditions can exacerbate the unreality of the situation (Ross and Morrison, 1989), although such artificiality can sometimes lessen in impact once participants become immersed in a task (Foster, 1996; Hammersley and Atkinson, 1995). Even when technical challenges have been overcome, researchers have many choices to make about what to video, whether naturalistic or experimental settings; individuals, pairs or groups; whether to focus on computer screen, facial expressions or hand gestures; and the like. There are also methodological challenges in relation to analysis. Coding the transcript can sometimes produce very different results to coding the video (Pirie, 1996), and different coders can sometimes code the same video data in quite different ways (Powell et al., 2003).

Ethical challenges: It is vital that researchers protect the rights and privacy of their participants (Hall, 2000; Mackay, 1988). In video-based research, anonymity is almost impossible to attain (Pirie, 1997) for studies in which, for example, participants' gestures and facial expressions are central. In presenting research findings by showing examples of data, changing the names of participants is not enough to assure them of their privacy. Alteration of voice and image may be required by participants. Pixilation and voice distortion are techniques easily employable for ensuring anonymity.

3.3 DIGITAL APPROACHES TO COLLECTING VIDEO DATA

This research is using learners' inferences (further discussion in section 4.3.) as a way of collecting externalised thought processing. Participants in this study think aloud and write information they extract when making inferences based on external mathematical representations shown to them on a computer screen. Recent developments in video technology are now offering opportunities to capture learners' interaction with computer-based representations.

3.3.1 RECENT DEVELOPMENTS IN VIDEO TECHNOLOGY

New developments in video technology have provided additional justifications, not just to use video in this study, but also to devise a new approach to capture learners' computer interactions. For example, portable digital video cameras have become widespread in developed countries. There is now also a variety of media available, such as digital video disk (DVD), external hard disks, memory cards and mini digital video tape. Meanwhile, the processing power of affordable desktop computers has increased to the extent that the manipulation of digital video is a viable option. Video is becoming more and more portable and affordable (PC WORLD, 2005). There are also new developments in video search technologies. For example, the beta version of Google Video has recently become available (http://video.google.com), which allows users to search closed captioning, text descriptions, and transcripts, and so jump straight to particular points in video clips. While automatic recognition of images in video clips is still fairly primitive in consumer systems, software to convert ordinary speech into text (i.e. automatic transcription), while not yet at the quality suitable for discourse analysis, is developing rapidly.

There are further notable developments that impact directly on this study: the typical desktop computer now has the power and capacity to log in video form, how users interact with it. A period of computer use – including all the screen activity, mouse movements and clicks, and key presses – can be recorded for later replay and analysis, without additional equipment. Moreover, the recent availability of unobtrusive eye-tracking and face-tracking devices allows the video capture of screen attention.

Even handwriting and sketching can now be recorded as live video through Tablet PCs, electronic whiteboards and small overhead mini-cameras.

3.3.2 NEW OPPORTUNITIES FOR RESEARCHING LEARNERS' COMPUTER INTERACTIONS WITH MULTIPLE REPRESENTATIONS

It can be argued that what learners see, what learners do with what they see, what they say, what they write down, and when they do all of this may provide some insight into their interactions with computers. However, traditional approaches to capturing these actions, particularly when multiple representations are involved, have been found to have some methodological constraints.

The 2003 study (also discussed in section 1.5) analysing video data of students problem-solving with multiple mathematical representations emphasises the need to identify which representation is being considered by a learner as utterances are made and to examine more closely students' movement between representations. However, when researchers study learners' sense-making of mathematical representations, verbal utterances are often not enough in isolation. For example, Ainley, Bills and Wilson (2004) recorded students' computer screen activity:

> ***Student:*** *You need times 'cause you need to that (points to 15 minutes) times twenty. (p. 6)*

Identifying exactly what the student is signifying by "that" is important for the analysis. Instances such as this occurred many times in the data that Ainley and her colleagues presented.

In principle, were researchers to recognize ambiguous signifiers at the time, they could clarify ambiguity by asking participants what they meant. This, however, can violate one of Ericsson and Simon's (1984) suggested guidelines which are intended to minimise disrupting participants' concentration whilst at the same time obtaining verbal data. Ericson and Simon suggest researchers should only intervene when participants stop talking and can only say "please keep on talking."

There are consequently numerous empirical studies using videos of learners' gestures such as pointing. However, participants do not always point to what they are referring. It is also typically difficult from these videos to pinpoint where learners are making inferences: numerical information for example might be extracted from displayed coordinates, from a graph, or from an algebraic expression.

Another example of where verbal utterances are not necessarily enough is when learners may prefer to think on paper (Villarreal, 2000). Pirie (1996) suggests complementing a think-aloud protocol with paper-based worksheets. However, when writing is analysed after the event, the temporal order of writing and the role of scratch work have to be guessed. Real-time records of learners' writing may provide important evidence such as erasure of previous inferences made. There are also mathematical terms that students may not express verbally, and writing or sketching may help them to describe what they want to say. However, without the opportunity to do this in real time, participants may not be able to provide crucial insights into their thinking.

The 2003 study that looked at learners' engagement with multiple representations, attempted to use video along these lines. In the study, four analogue video cameras were used to record a pair of students' interactions with a computer and a paper-based worksheet. The aim was to gain an understanding of participants' learning from multiple representations, particularly mathematical representations. The data collected were sent to be professionally edited to combine all four videos in a "quad image" (i.e. showing all videos simultaneously in one screen). This was then transcribed using a word processor. Although this approach allowed what students do, say, see and write to be captured, the disadvantages described earlier were evident. Due to the limitation of visual range and learners' movement, it was difficult to see the totality of the process. In particular, the reduced size of each video stream made it difficult to see what was going on; the writing and screen could occasionally become obscured by the student's body; it was typically not possible to tell exactly where on the screen the student's attention was focused; the task of coding the data in detail was laborious; and transcripts of talk, gestures and writing could not be easily combined.

In Gluck's study (1999), the worksheet was made available in the computer screen. Participants of Gluck's study were asked to type their answers. He investigated students' eye-movement during problem solving. This type of study, however, constrained students to sketch. Also, students used a 'chin rest' to minimise head movement during problem solving. Such an approach could be argued as 'unnatural' and intrusive.

Over a number of years, others have developed an approach to the study of users' interactions with computer systems for the purpose of learning. Until now, no study has attempted to coordinate video recording of action and utterances, eye movement recording to track the focus of the user's attention on the screen, and handwriting recording to look at what users are writing.

3.3.3 INNOVATIVE APPROACHES TO CAPTURING WHAT LEARNERS SEE, SAY, WRITE AND DO

Using the latest digital technologies, a set of techniques has been developed to capture, coordinate and analyse learners' gazes, real-time writing, utterances and actions (further discussion on the coordination and analysis of such data is in chapter 5). This section illustrates the analytical advantages of using this technique.

People's perceptions of visual information are important factors in studying computer interaction (Hansen, 1991). To identify something what a person is seeing is possible using eye-tracking cameras. Moreover, eye tracking records can clarify ambiguous subjects that can be found within participants' verbal data such as "this..." and "it..." (Hansen et al., 2001). However, computer interaction cannot solely be judged on the basis of eye-tracking data because, for example, some information can be picked up by peripheral vision (Hansen, 1991). Participants' attention can also drift, or participants can happen to be staring at a computer screen whilst organising their thoughts. Eye tracking therefore complements but does not replace the recording of utterances and gestures.

As mentioned in the previous section 3.3.2, some learners prefer to think on paper (Villarreal, 2000), and so written work can sometimes provide insights when combined with video (Pirie, 1996). A protocol that allows learners to write or sketch while they work can feel both more natural to the participants and provide greater insight into their thinking.

Building on this experience, and on the suggestions of Ericsson and Simon (1984), Pirie (1996) and Hansen et al. (2001), a set of techniques amalgamating eye-tracking, screen capture, Tablet PC and digital video technology to capture what people see, say, do and write when interacting with computers has been developed and refined.

To gain insight into what a person is looking at is possible using an eye-tracking device (Hansen et al., 2001). This can also be coupled with real-time writing using Tablet PC screen capture. Figure 3-1 provides an example of what can be collected using these technologies. The two figures on the right are both 'screen activity'. The upper-right is what the observer sees during the study. The lower-right figure is an image generated by the analysis software showing where the eye dwelled on an element of the screen. Further discussion of this is done in the next section. By superimposing eye gazes on the screen, the researcher can clearly see shifts in attention.

The next section describes this data capture setup.

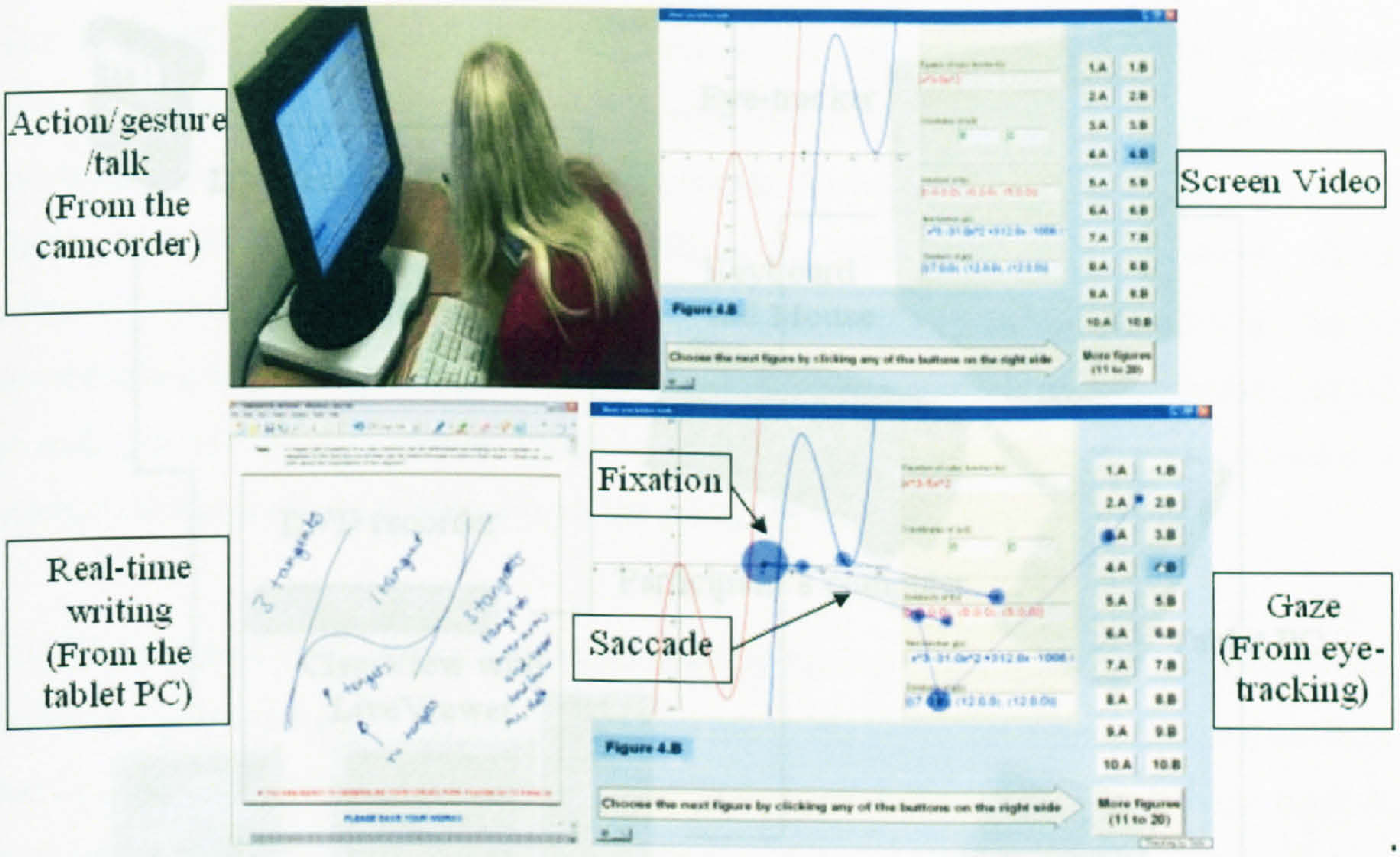


Figure 3-1 Examples of action, writing, screen and gaze videos

3.4 DATA CAPTURE SETUP

The key parts of the data capture hardware setup (Figure 3-2) are a digital video camera to record participants' actions and utterances, a non-intrusive eye-tracking device (the particular device chosen was the Tobii ET-1750™) with screen capture to record gaze and software event logs; and a Tablet PC with screen capture to record handwriting. The setup is designed to capture what participants are seeing on the screen, what they are saying about what they are seeing, where in particular they are looking on the screen, what they are doing (e.g. pointing at the screen, gestures, mouse movements, off-screen attention), and what they are writing.

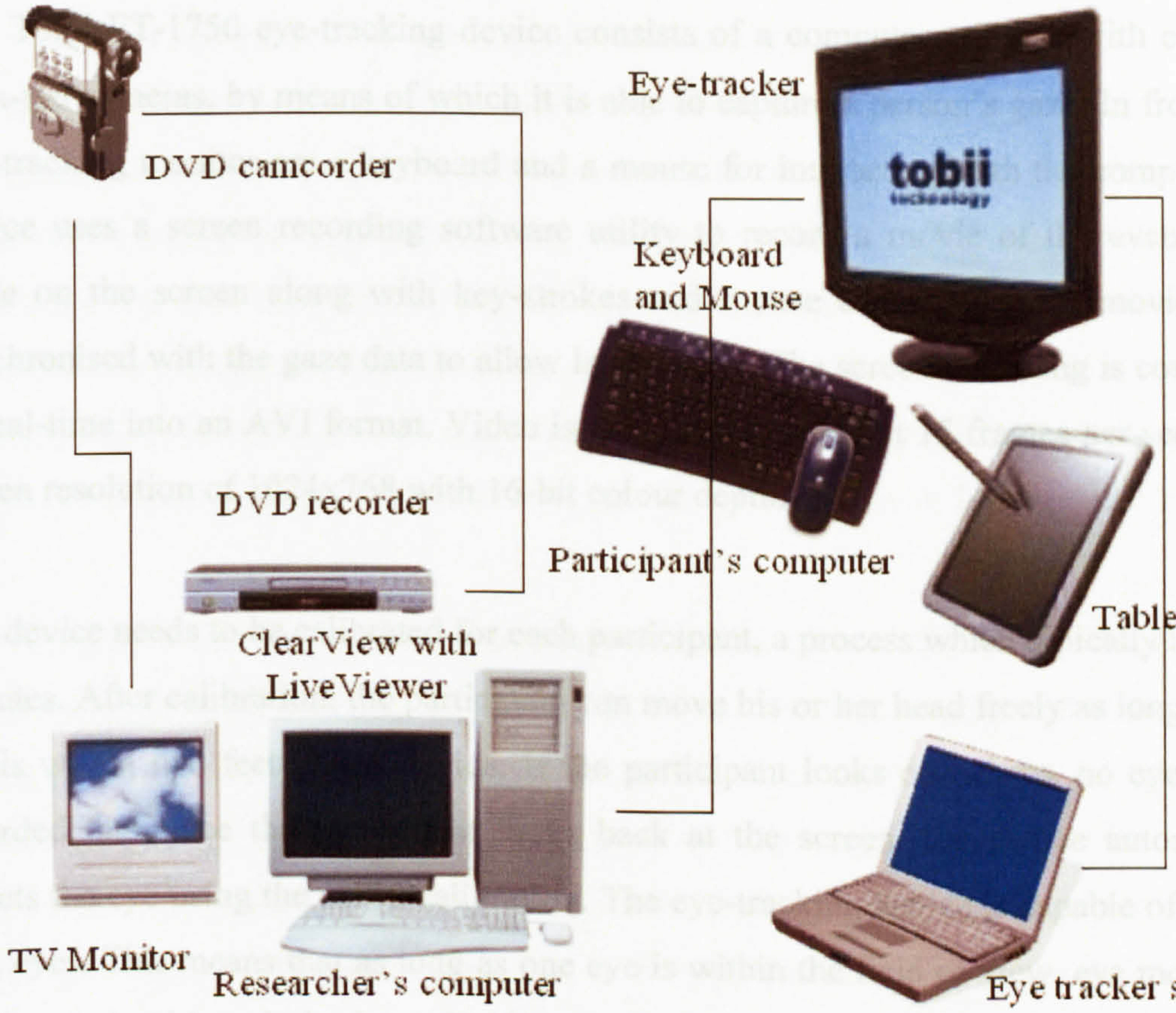


Figure 3-2 The data capture setup

A mini DV-camcorder with a directional shotgun microphone is mounted on a tripod positioned at the side of the participant. The camera is connected to a DVD recorder which automatically records what the camera captured directly onto a DVD instead of a videotape. This helps in organising data since memory capacity means it would not be possible to store all episodes on the computer, given the high screen resolutions needed to identify precisely at what symbols or parts of a curve, participants might be pointing. Moreover, backup copies are easier to make. This digital video can then be processed directly into an MPEG file playable using a wide range of analysis software.

Portability of this setup would be improved by using a camera that can record directly to DVD instead of a separate DVD recorder; however available models were limited in the length of recordings they supported.

The Tobii ET-1750 eye-tracking device consists of a computer monitor with embedded infra-red cameras, by means of which it is able to capture a person's gaze. In front of the eye-tracking monitor are a keyboard and a mouse for interacting with the computer. The device uses a screen recording software utility to record a movie of the events taking place on the screen along with key-strokes and mouse clicks, and this movie can be synchronised with the gaze data to allow later replay. The screen recording is compressed in real-time into an AVI format. Video is typically captured at 15 frames per second at a screen resolution of 1024x768 with 16-bit colour depth.

The device needs to be calibrated for each participant, a process which typically takes two minutes. After calibration, the participant can move his or her head freely as long as he or she is within two feet of the device. If the participant looks elsewhere, no eye mark is recorded, but once the participant looks back at the screen, the device automatically detects the eye using the saved calibration. The eye-tracking device is capable of tracking both eyes. This means that as long as one eye is within the field of view, eye movements are detected. Although the latest Tobii eye-tracking device is able to track most people's eyes, difficulties were observed in tracking people who need to wear glasses when working with a computer[1]. At present, attempts to improve eye detection are being undertaken. The device also cannot tell if the eyes are unfocused. The previous version of the Tobii eye-tracking device is not advisable because it is unable to track the pupils of people with dark brown eyes.

On the desk is a Tablet PC, positioned according to whether a participant is left handed or right handed, where the participant can make notes. A Tablet PC is essentially a flat-panel portable PC that allows the use of a stylus to input data by tapping and directly writing on the screen. One can sketch, draw, and erase in a similar way to using pen and paper, although clearly the sensation is somewhat different, and can therefore feel artificial. In trial studies, participants unfamiliar with Tablet PCs became comfortable with the device after practising for a few minutes. Camtasia™ screen capture software is used to capture events on the Tablet PC.

[1] At the time of writing, an update in the software, designed to overcome this particular problem, has just become available.

The researcher sits behind the participant where another computer is setup to control the eye-tracking device and a monitor shows the camcorder output. This computer is also showing a live stream of what the participant sees and does on his or her machine with the participant's eye-gaze superimposed on the screen (this is called a 'LiveViewer' in ClearView™). Watching this live screen record and the camcorder monitor helps the researcher to formulate questions for any subsequent interview, and to act if the participant's eyes move out of the field of the eye-tracking camera's view. This can happen when participants immersed in the task cover their eyes or lean on their hands.

It is recognised that in conducting qualitative studies of realistic educational behaviour, the phenomena being investigated should, as far as possible, be observed in their natural setting (e.g. Hammersley and Atkinson, 1995). The setup described here, though, requires a contrived setting, primarily to ensure accurate eye-tracking. For example, the laboratory has controlled lighting to minimise effects that might affect the collection of eye gaze; and the participant's chair has no wheels, in order to minimise movement that would take the participant more than the two foot maximum limit allowed by the eye-tracking device.

There are three types of video data captured: the 'gaze video', the 'action video', and the 'writing video'. Each of these types is now described.

3.4.1 THE GAZE VIDEO CAPTURE

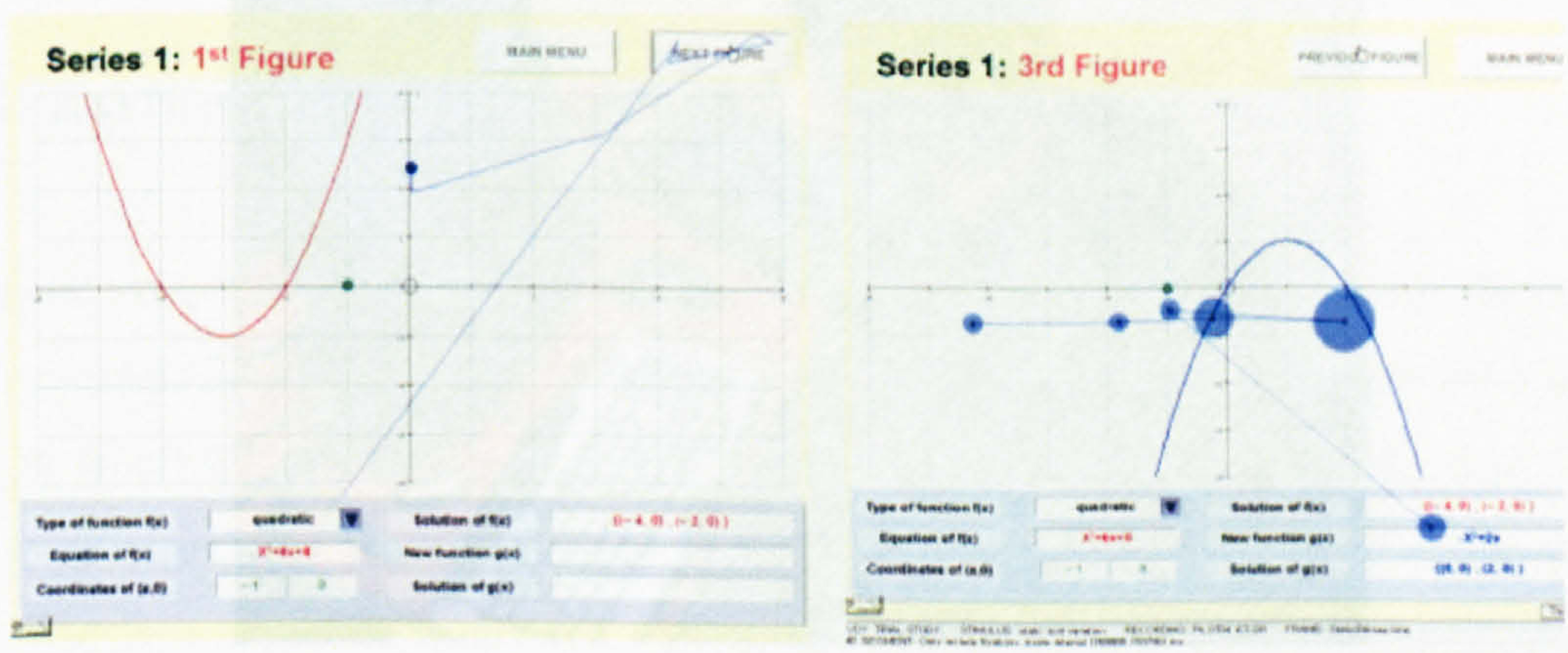


Figure 3-3 Example snapshots of gaze video

The **'Gaze video'** shows the screen contents of participants' interaction with the software, with the gaze replay superimposed. In Figure 3-3, the blue lines indicating the path that the eyes took across the screen are called 'saccades'; the blue circles show where the eye dwelled on an element of the screen for a length of time above a specified threshold, and are called 'fixations'. A variation in fixation threshold can result in qualitatively different fixation patterns. A typical threshold is 100 milliseconds as used in other studies about representations (see e.g. Corte, Verschaffel and Pauwels, 1990, Knoblich, Ohlsson and Raney, 2001). It is also possible for the length of a fixation to be indicated by the size of the circle. Another advantage is that this video can be used for retrospective interview.

3.4.2 THE ACTION VIDEO CAPTURE

'Action video' (Figure 3-4) – a playback of the video and audio record captured by the digital camcorder. Some of the actions that can be captured are gestures, head movement from the eye-tracking screen to the Tablet PC or to somewhere else, typing, use of mouse, and utterances. A single camera can only focus at a certain angle and the camera's position determines what can be captured. However, the researcher can decide what is vital to the investigation at hand. For example, camera angle of the figure below cannot capture participants' facial expression.

Figure 3-4 Snapshot from an action video

3.4.3 THE WRITING VIDEO CAPTURE

'Writing video' (Figure 3-5) – a recording of Tablet PC operation such as handwriting, sketching, drawing, erasing and flipping from one page to another. There are situations where learners use a pen to sketch something without actually marking anything on paper. Use of the Tablet PC captures this 'virtual sketching' because the 'pen-pointer' movement represented by a black dot is recorded.

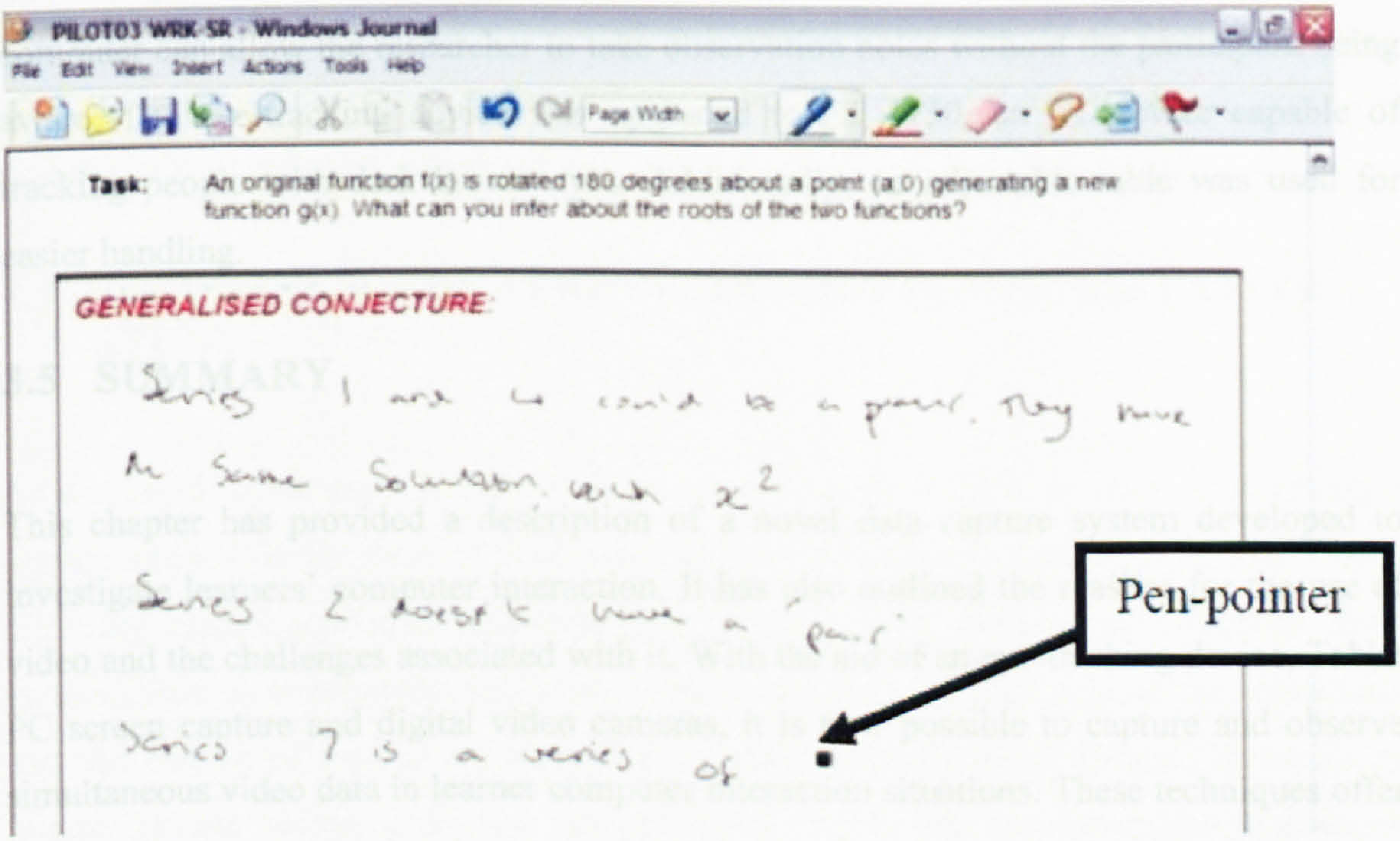


Figure 3-5 Snapshot of a writing video

3.4.4 TESTING THE DATA CAPTURE SETUP

A test-run was conducted with two pilot participants. During the time of the testing, the only eye-tracking device available was the first model of the Tobii eye-tracking device (ET-17). Tobii ET-17 had been found to have difficulty tracking people with dark brown eyes. In this test run, the following practical challenges were addressed. Equipment was positioned in a way that it ensured that 1) the camera records the best visual range of recording 'action video' possible 2) the position of the Tablet PC can be comfortable for participants to write on, without screen glare (i.e. an improvised docking device was used) 3) the eye-tracking device sits steadily to avoid loss of calibration and eye-tracking capture 4) appropriate lighting for the ET infra-red cameras to avoid refractions and also avoiding natural infra-red light from the sun 5) there is enough space on the table for accommodating left or right-handed participants 6) the height and position of table and chair can be adjusted easily (i.e. both the table and the chair used have a lever to adjust height and enough room to move the chair backward or forward) 7) the recording follows a certain sequence which is the digital video camera followed by the Tablet PC capture then the eye-tracking for later synchronisation purposes 8) the position of the researcher's computer can allow the researcher to take observation notes without the participant being aware. The eye-tracking device was upgraded to ET 1750, an ET device capable of tracking people with dark brown eyes. Additionally, an adjustable table was used for easier handling.

3.5 SUMMARY

This chapter has provided a description of a novel data capture system developed to investigate learners' computer interaction. It has also outlined the reasons for the use of video and the challenges associated with it. With the aid of an eye-tracking device, Tablet PC screen capture and digital video cameras, it is now possible to capture and observe simultaneous video data in learner computer interaction situations. These techniques offer potentially interesting developments in understanding learners' computer interaction particularly in the context of learners' engagement with computer-based representations.

The next chapter discusses the application of this approach to a particular study of learning from multiple representations.

4 CAPTURING LEARNERS' UTTERANCES AND ACTIONS WITH MULTIPLE REPRESENTATIONS

4.1 INTRODUCTION

This chapter provides a rationale for the representations selected and for the quasi-experimental design adopted to investigate learners' completion of three comparable tasks about computer-based mathematical representations which represent the main study of the thesis. It shows how the data capture system described in the previous chapter was used to collect learners' interaction with multiple representations. This chapter explains the design of the experiment conducted.

4.2 ADOPTING A ROTATIONAL DESIGN IN A QUALITATIVE STUDY

As discussed in section 2.2, learners have often found it difficult to learn from multiple representations. In the case of mathematical representations, for example, learners have difficulty relating equivalent representations such as graphs, numbers and equations. Such mathematical representations can now be easily accessed and manipulated via computers. Yet the effects of having representations instantiated differently (Static, Dynamic or Interactive) are unclear. Although there are some studies that have attempted to research the differences of a combination of either Static versus Dynamic (e.g. Lowe, 2003), or Dynamic versus Interactive (e.g. Bodemer et al., 2004), or Static versus Interactive (e.g. Evans and Gibbons, 2006, in press), no study has been identified that investigated these three types of instantiations in one study.

The integrated capture of learners' gesture, speech, object of attention, and handwriting can offer the opportunity of identifying a range of strategies for using multiple representations. Since representations are one of the bases for making inferences (Larkin and Simon, 1987), this research used learners' inferences as a way of collecting externalised thought processing. Learners' inferences can be a combination of verbal, graphical and mathematical forms (see e.g. San Diego, 2003). The author decided that a qualitative approach is deemed apposite for investigating how learners use multiple representations. In order to identify learners' strategies, it is valuable to observe the processes during a problem solving task (Aczel, 1998). However, there are situations in which qualitative data can yield quantitative data (e.g. Chi, 1997; Powell et al., 2003; Savenye and Robinson, 1996). Frequencies of learners using the same strategies can be counted and presented using descriptive statistics.

Investigating the effects of varying representational instantiations on learners' strategies can be complex, particularly if it is to involve three types of representations and three types of instantiations. For example, the instantiations of a preceding task or the nature of the task itself may influence the way a learner approaches a subsequent task (e.g. Ainsworth, 2006). An appropriate way to reduce this complexity is by using a quasi-experimental design.

Table 4-1 The rotational design adopted for assigning instantiations

Single-instantiation	S→S→S	Varying-instantiations	S→D→I
			S→I→D
	D→D→D		D→S→I
			D→I→S
	I→I→I		I→S→D
			I→D→S
(Where: S – Static, D – Dynamic, I – Interactive)			

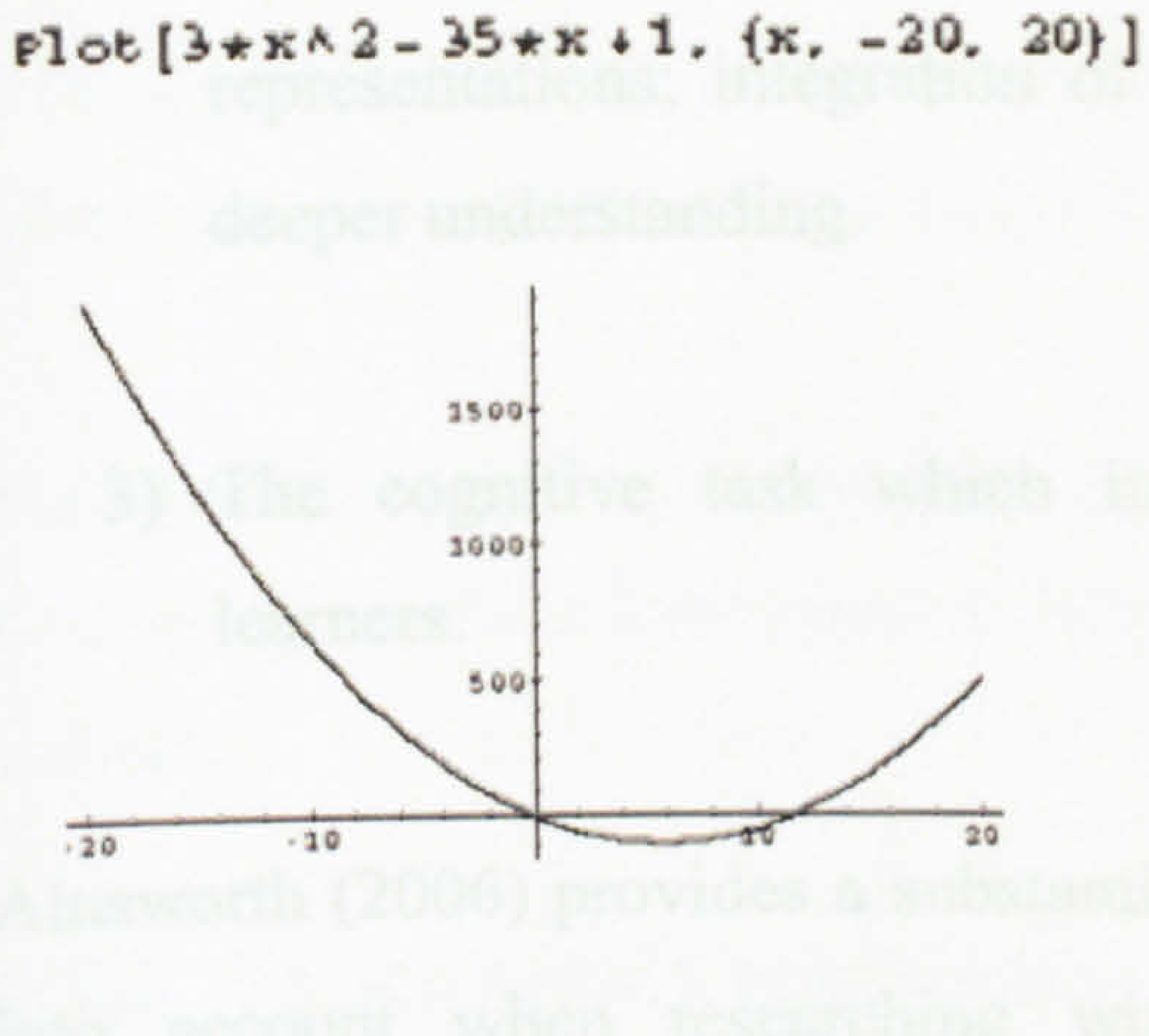
A rotational design was adopted in assigning the order of the instantiations. This was done to reduce 'carry over effects' caused by the order of the instantiations given to the participants. There are six possible ways of rotating the sequence of three types of instantiations. There are also three ways to pair three tasks for each form of instantiation. To minimise the effect of a subsequent task, the pairing of a task with an instantiation was varied randomly. The rotational design is given in Table 4-1 above. By comparing the data derived from participants involved in tasks having only one form of instantiation to those from participants receiving different types of instantiations, this design hopes to lessen the possibility of the results being attributable to the sequence of the task and the sequence of the instantiation.

Apart from the assignment of tasks and sequence of instantiations, specific details of the tasks that were used and the design of the instantiations, were also considered. These are discussed in the next section.

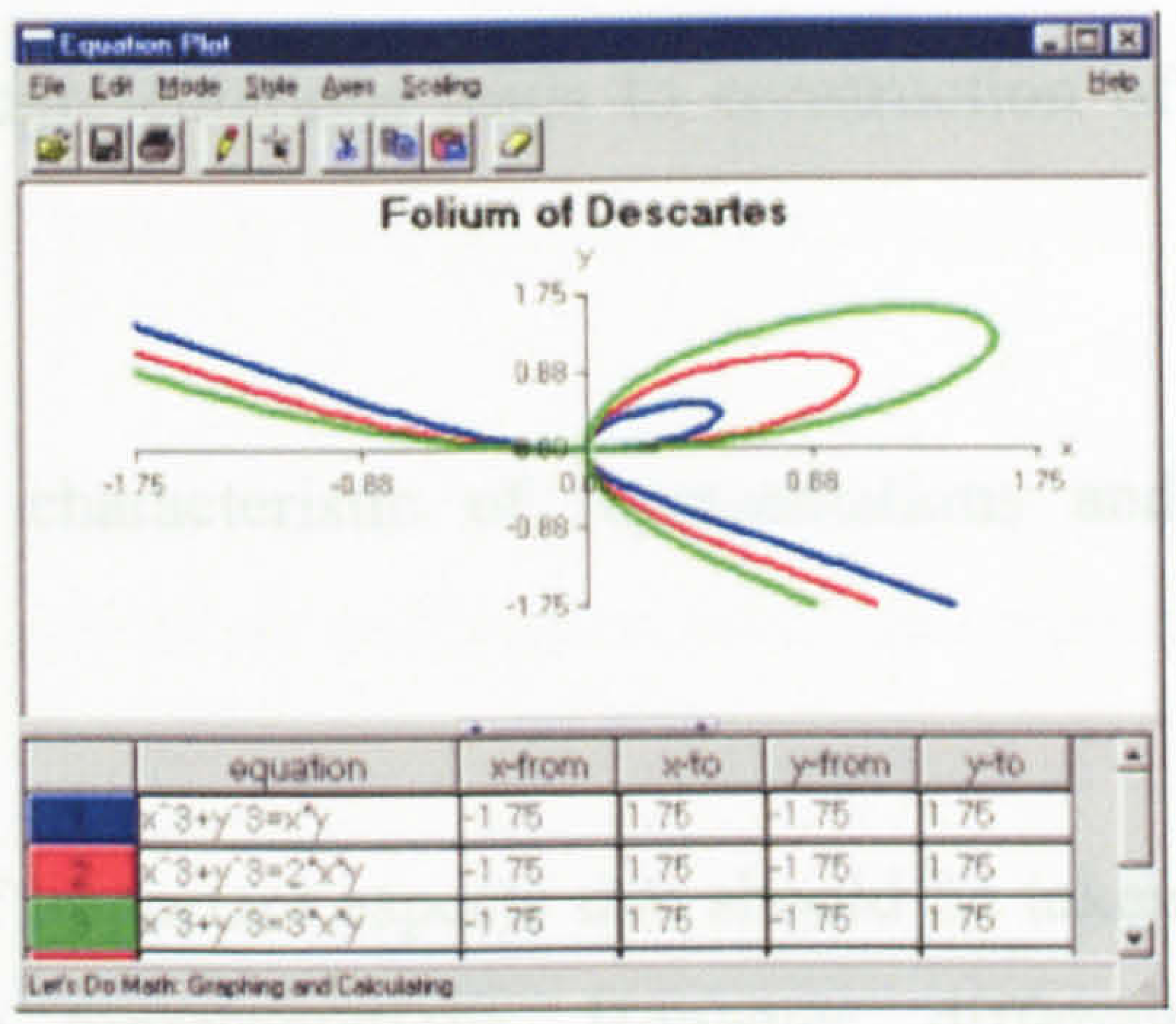
4.3 THE DEVELOPMENT OF THE TASKS

Section 2.2 shows that learners' difficulties linking multiple representations have included mathematical functions (i.e. graphic, numeric and algebraic representations). Examples of mathematical representations presented to learners in varied instantiations (Static, Dynamic and Interactive) have been widely available via the internet and commercially (Figure 4-1 below). The designs of mathematical computer-based representations were evaluated and considered in this study. The conflicting claims as presented in chapter 2 related to how mathematical representations are instantiated needs further research. The design of the representations, the task learners undertake, and learners' prior knowledge interact and provide complexity in researching learners' interaction with computer-based representations (Ainsworth, 2006; Cox, 1999).

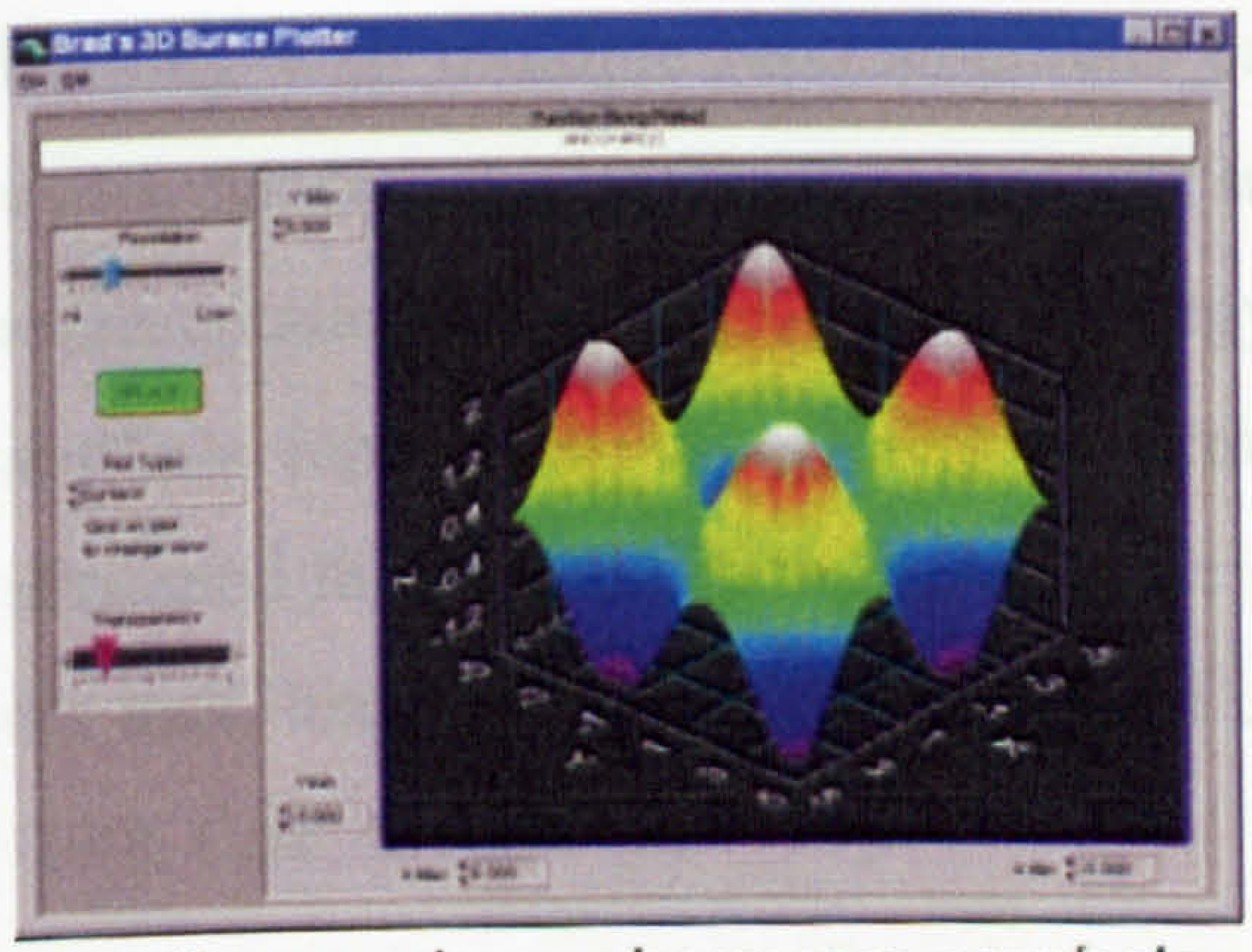
Multiple representations can be helpful when people are dealing with complex information. Ainsworth (2006) proposes a framework for identifying factors influencing the effectiveness of multiple representations. The framework for learning with multiple representations suggests considering three fundamental aspects of learning: the design of multiple representations, the functions of multiple representations, and the tasks. The DeFT (Design, Functions, Tasks) framework for learning with multiple representations proposes that the following need to be considered:



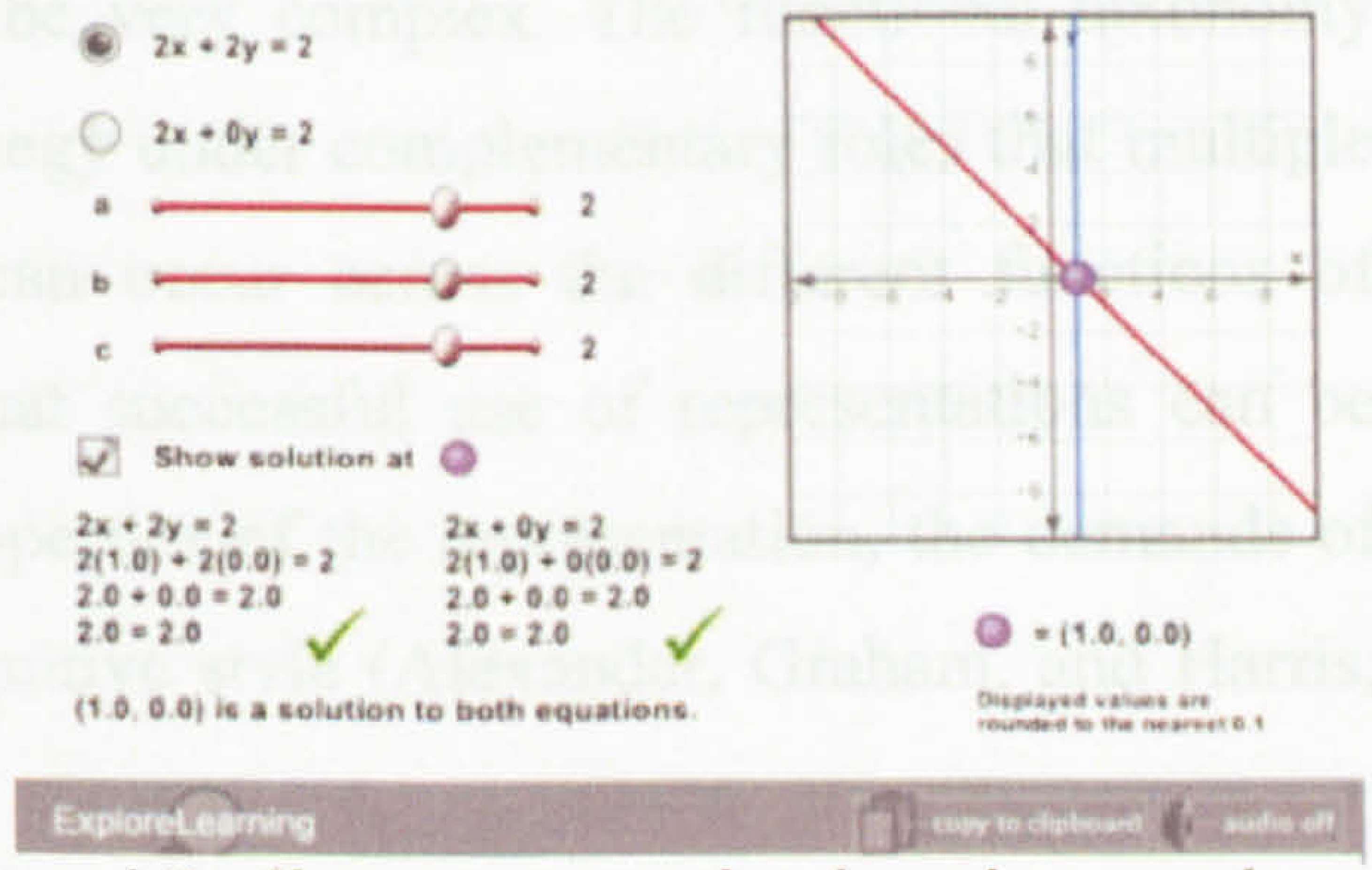
www.cs.ucsd.edu/.../Mathematica/Example/examples.html



www.mathresources.com/products/graphing/tour4.html



http://www.scienceshareware.com/calc-3dplotter.htm



http://www.mcs.explorelearning.com/.../P1000/Page.htm

Figure 4-1 Examples of mathematical representations presented in various instantiations

1) The dimensions of designs: number of representations; information that each representation conveys; combination of external representations and its instantiation; sequence of presentation or construction of representations; translation as indicated by the nature of instantiation.

2) The functions that multiple representations serve (see Figure 4-2 Functions of MERs taken from Ainsworth, 1999, p. 134): representations differ in the processes each supports or in the information each contains, and these may complement each other; certain representations constrain interpretation of another

representations; integration of different representations leads to construction of deeper understanding.

3) The cognitive task which impacts the characteristic of representations and learners.

Ainsworth (2006) provides a substantial report of important aspects that should be taken into account when researching with multiple representations. However, different techniques to identify differences between the effect of the design, task, and function in learning with multiple representations can be very complex. The functional taxonomy which Ainsworth (1999) proposes, puts strategy under complementary roles that multiple representations serve. However, strategy can occur across the different functions of multiple representations. Others confirm that successful use of representations can be attributed to the interaction between the properties of the representation, the demands of the task, learners' prior knowledge and cognitive style (Alexander, Graham, and Harris, 1998; Cox, 1996; Cox, 1999).

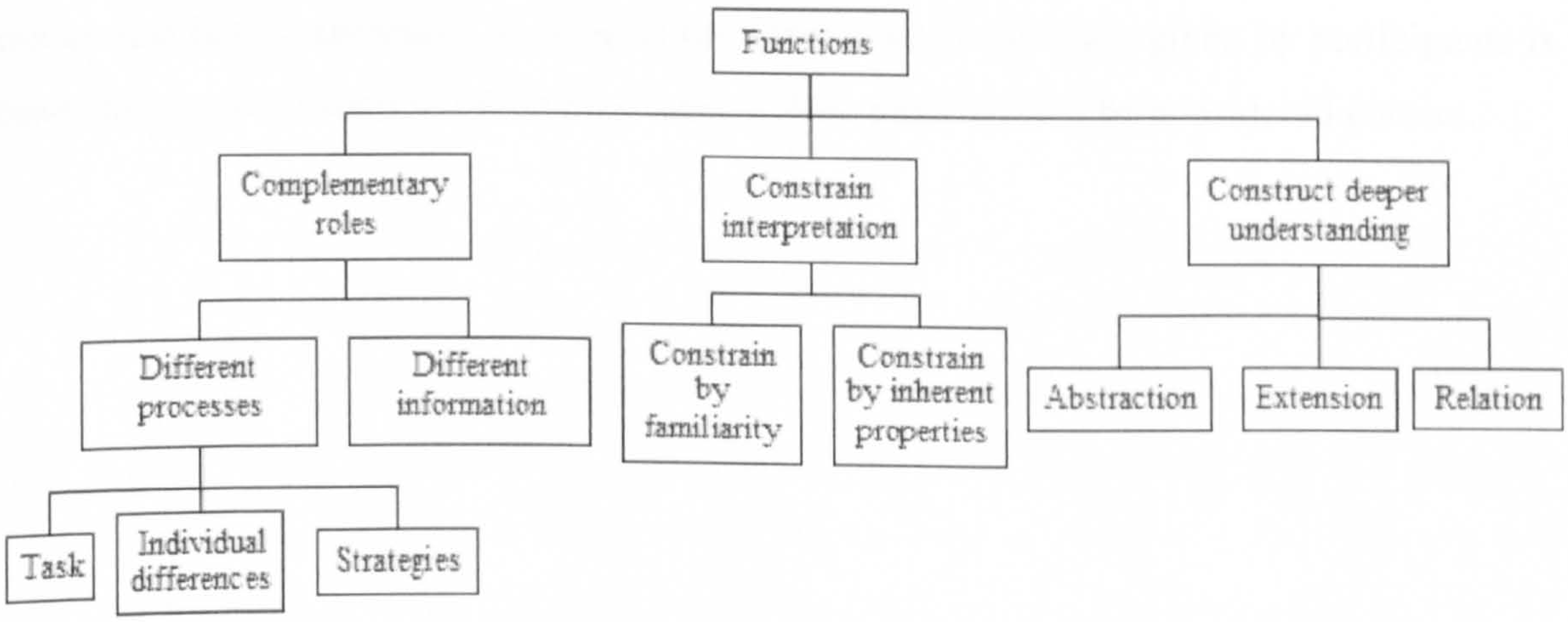


Figure 4-2 Functions of MERs

Ainsworth suggestions of a framework for analysing the effectiveness of multiple representations were taken into account. However, the gap identified in the literature in relation to learners' use of strategy with varying instantiation seems to suggest that strategy in Ainsworth's framework is understated. This study sets out a case that an alternative appropriate unit in studying the effect of varying instantiations is strategy use.

Mathematical representations were chosen as the domain to study the effect on learners' strategy of varying instantiations because of the persistent difficulties identified in the literature.

4.3.1 TASKS ABOUT MATHEMATICAL REPRESENTATIONS

The task in the study should not be familiar to the participants but achievable to a person with A-level Maths qualification. The tasks 1) were only about cubic functions to control for the inclusion of more representations. 2) can be completed either using one form of external mathematical representation or a combination of two or three – table of values, graphs, and equations 3) were to make inferences on external mathematical representations of cubic functions. The tasks and their corresponding expected generalised conjectures are presented below (Table 4-2). The generalised conjectures that participants may give can be different to the ones given below or may even be expressed not as text but by sketches. As long as the generalised conjecture given by participants is equivalent to the meaning of the ones below, their answers will be considered correct.

Table 4-2 The tasks

Task Name	Task	Expected generalised conjecture
Root	An original function f(x) is rotated 180 degrees about a point (a, 0) generating a new function g(x). What can you infer about the roots of the two functions?	The new roots are reflections of the roots of the original. The same distance from point (a, 0).
Chord	A chord can be drawn on the curve of a cubic function. What can you infer about the midpoints of the chords drawn between two points on any cubic function?	It is a set of all points bounded by the axis and the cubic itself.
Tangent	Any smooth function curve has regions (set of points) that can be determined according to the number of tangents to the function curve that can be drawn through a point in that region. What can you infer about the boundaries of the regions?	The regions are bounded by the curve itself and the inflection tangents (tangents through point of inflection) where the tangents change direction as the point of tangency moves along the curve.

These tasks[2] were developed in three different instantiations (Static, Dynamic and Interactive).

4.3.2 THE REPRESENTATIONS AND INSTANTIATIONS

The specific multiple external mathematical representations used in each task were carefully considered and expert advice was sought on decisions made regarding the representations to include, and the way they were visually presented on the interface. For example, the graphic representations are all 2-D Cartesian graphs (drawn on a XY plane); the numeric representations were coordinates of relevant points; and the algebraic representations are equations of graphs specific to the tasks. Moreover, the mathematical representations used for each of the tasks were more or less the same in number. Also, as shown in Table 4-3, each of the graphical representation has a corresponding numerical or algebraic equivalent.

[2] The author especially acknowledges Prof. John Mason of the Centre of Maths Education, The Open University for allowing the use of these tasks in this PhD project and for his helpful advice.

Table 4-3 The mathematical representations involved in each task

Task	Representations		
	Algebraic	Graphic	Numeric
Root task	• Equation of f(x) • Equation of g(x)	• Graph of f(x) • Graph of g(x) • Solutions of f(x) • Solutions of g(x) • Point (a, 0)	• Solutions of f(x) • Solutions of g(x) • Coordinate of (a,0)
Chord Task	• Equation of f(x) • Equation of the axis of symmetry	• Graph of f(x) • Midpoint • Chord • Axis of symmetry	• Coordinates of the endpoints of a chord on f(x) • Coordinate of the midpoint of a chord
Tangent task	• Equation of f(x) • Equations of the tangent lines	• Graph of f(x) • Tangent lines • Point not on f(x) • Point/s of tangency	• Coordinate of the point not on f(x) • Point/s of tangency

4.3.3 THE SOFTWARE

The types of instantiations are defined in this thesis as follows: "Static" (non-moving, non-changing, non-Interactive), "Dynamic" (capable of animation through alphanumeric inputs), and "Interactive" (directly manipulable graphs). These definitions served as the first guideline in designing the interfaces of the learning software. The author designed the Static interface interfaces first. Then, the Dynamic and Interactive versions were designed based on the interfaces of the Static versions. To some extent, the interface design is orthogonal between instantiations.

Suggestions to reduce difficulties associated with cognitive overload (see e.g. Sweller and Chandler, 1991) and attention (see e.g. Mayer and Moreno, 1998) was considered in designing the interfaces of the software. The design elements that were made consistent across the different types of software developed were the colours, position and size of the mathematical representations, 'input text boxes', buttons and slider. Also, the visual appearances of external mathematical representations were created similar to existing graphing software available via internet or commercial (e.g. Graphmatica, Autograph, GeoGebra, Cabri Geometry, Geometer's Sketchpad, Calc 3D, etc.).

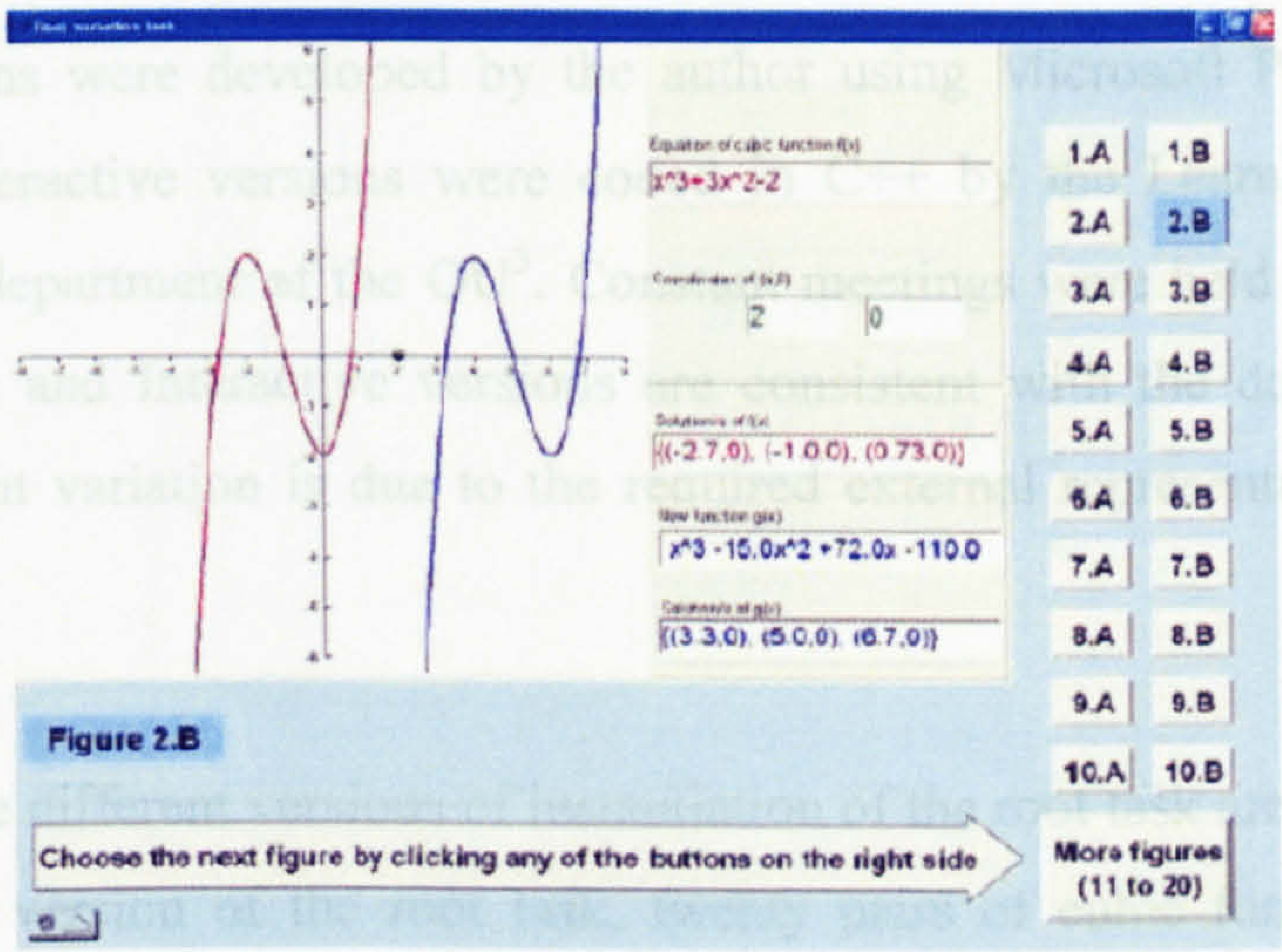


Figure 4-3 a) Static Root task (SR)

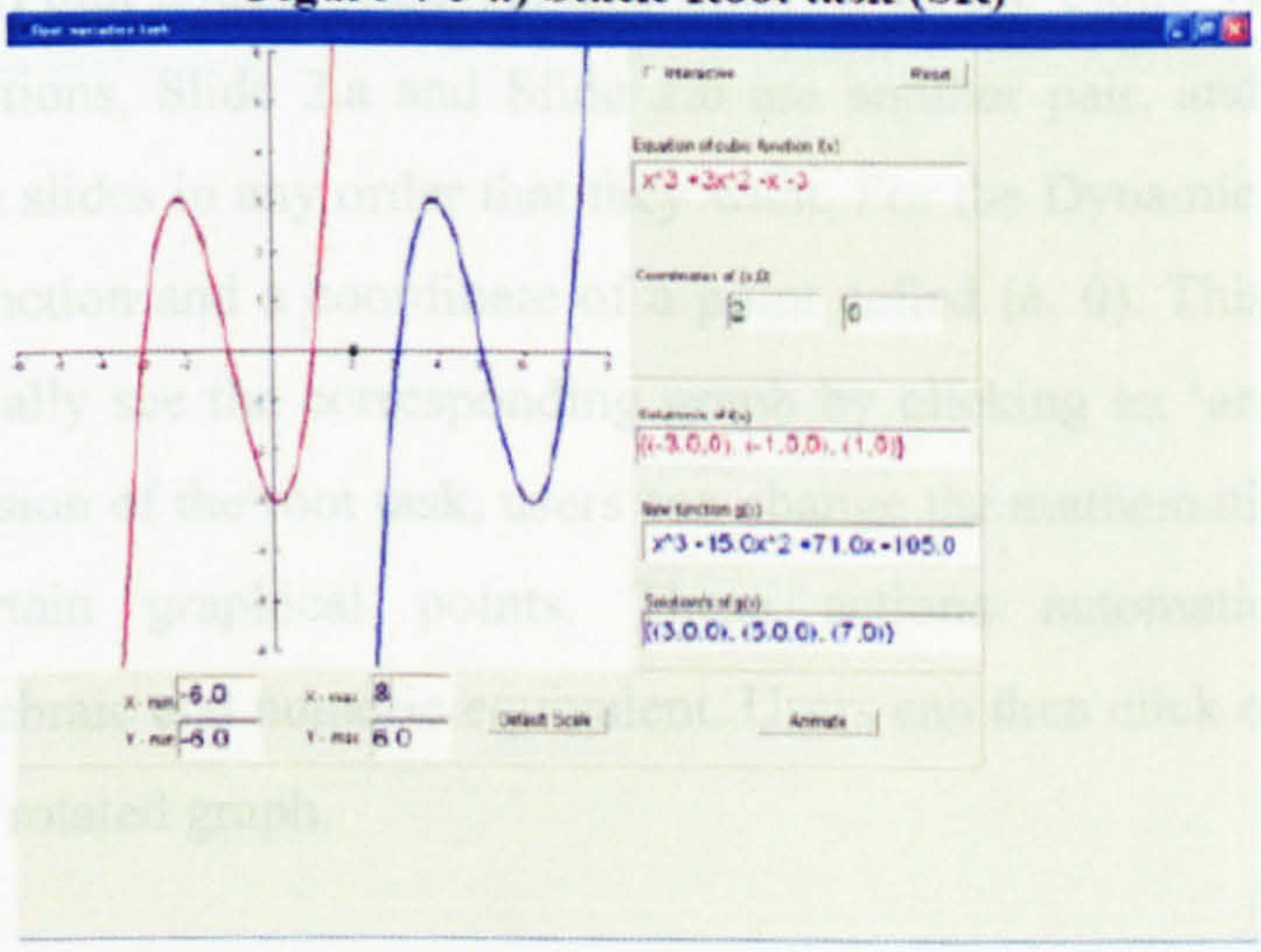


Figure 4-3 b) Dynamic Root task (DR)

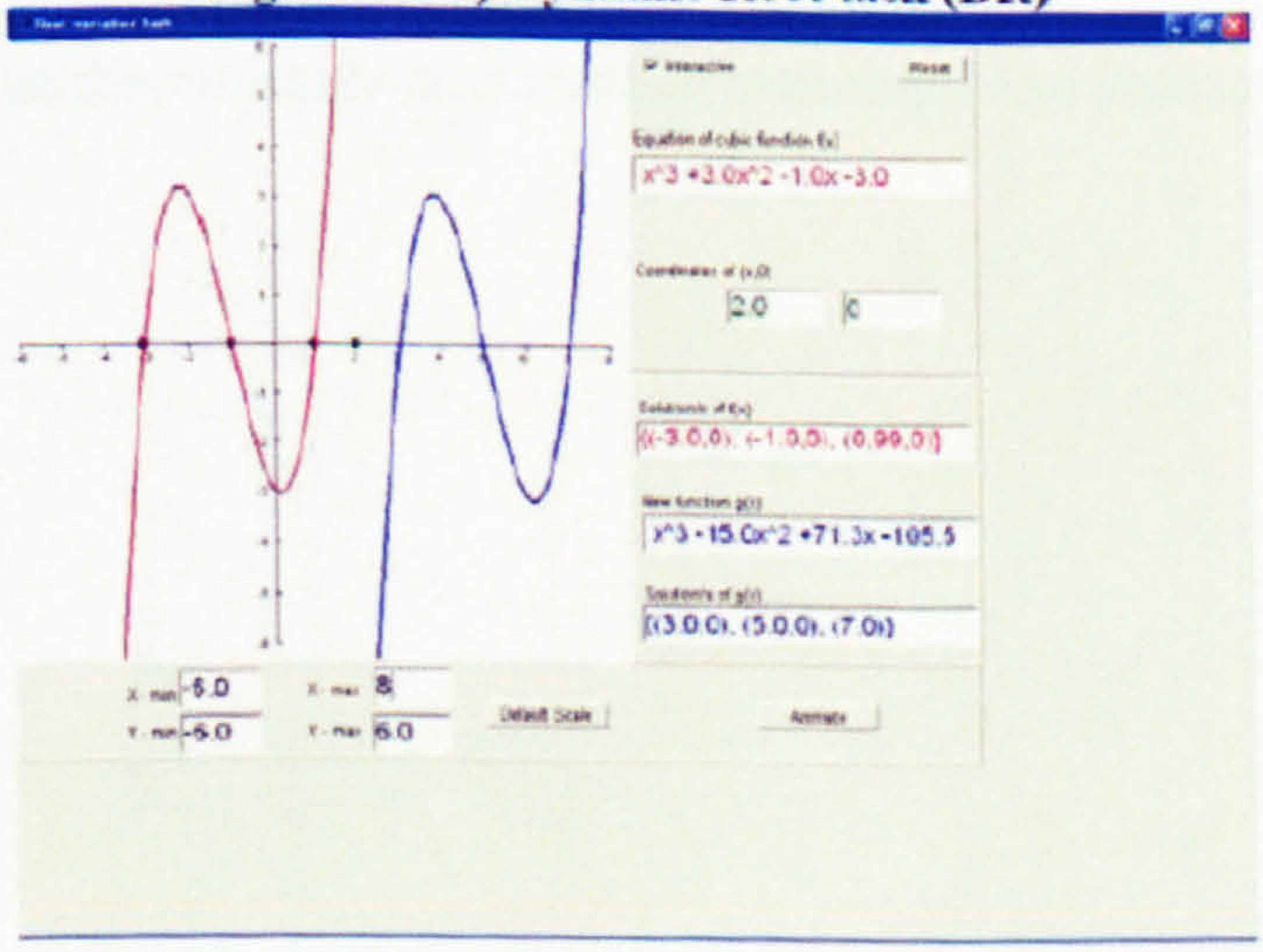


Figure 4-3 c) Interactive Root task (IR)

Figure 4-3 Instantiation of the Root task

The Static versions were developed by the author using Microsoft PowerPoint™. The Dynamic and Interactive versions were coded in C++ by the Learning and Teaching Solutions (LTS) department of the OU[3]. Constant meetings were held in order to ensure that the Dynamic and Interactive versions are consistent with the design of the Static version. The slight variation is due to the required external representations and buttons specific to a task.

Screenshots of the different versions of instantiation of the root task are presented in a, b, c. For the Static version of the root task, twenty pairs of cubic functions were made available in PowerPoint slides. Slide 1.a and Slide1.b of the Static versions represent a pair of cubic functions, Slide 2.a and Slide 2.b are another pair, and so on. Users can choose to view the slides in any order that they want. For the Dynamic version, users can enter the cubic function and a coordinate of a point called (a, 0). This action allows the users to automatically see the corresponding graph by clicking an 'animate' button. For the Interactive version of the root task, users can change the mathematical representations by dragging certain graphical points. These actions automatically change the corresponding algebraic and numeric equivalent. Users can then click on "animate" to see the corresponding rotated graph.

[3] Thanks to Mr. Collin Thomas and Mr. Geoff Austin for implementing the software in C++.

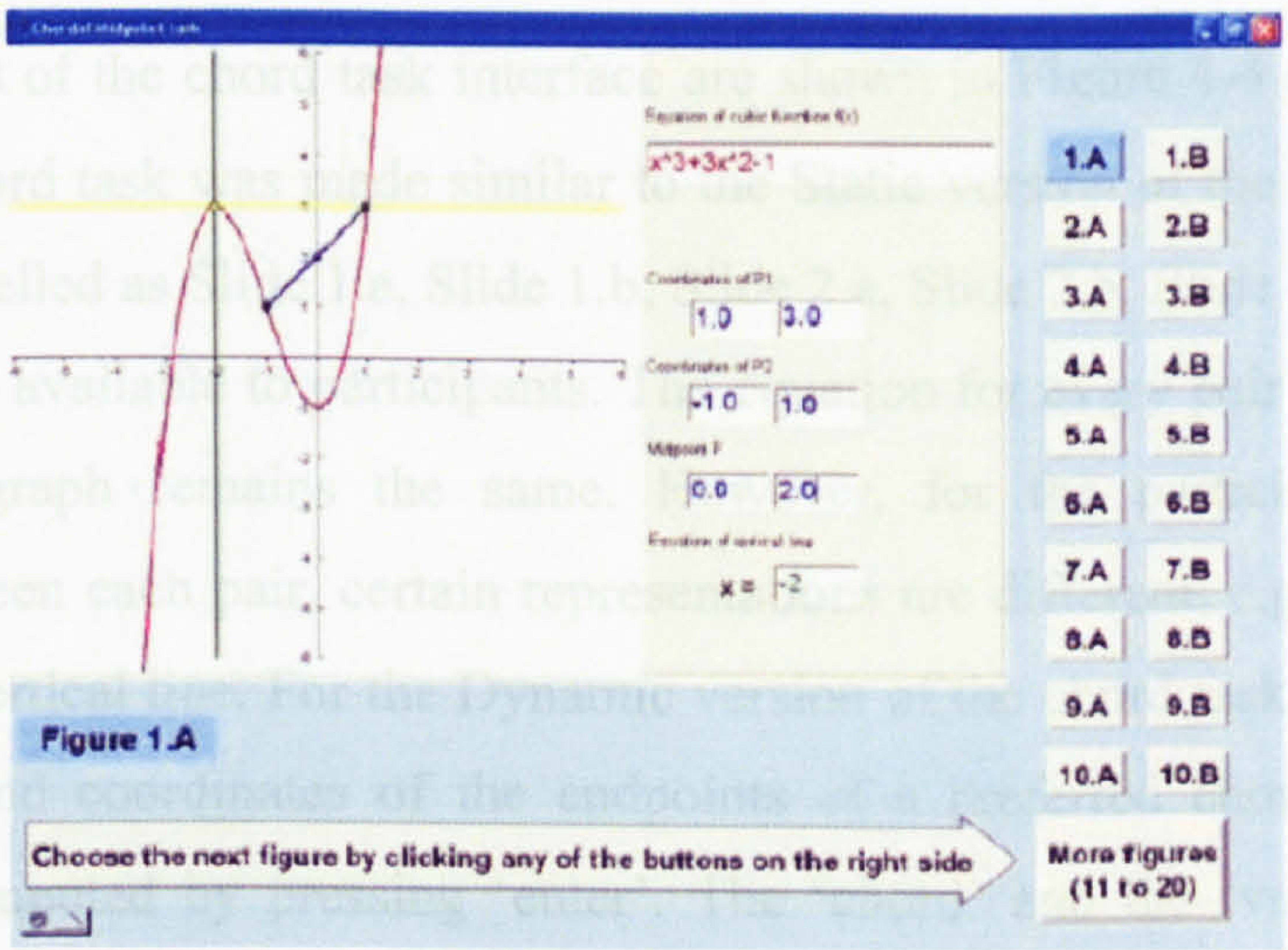


Figure 4-4 a) Static Chord task

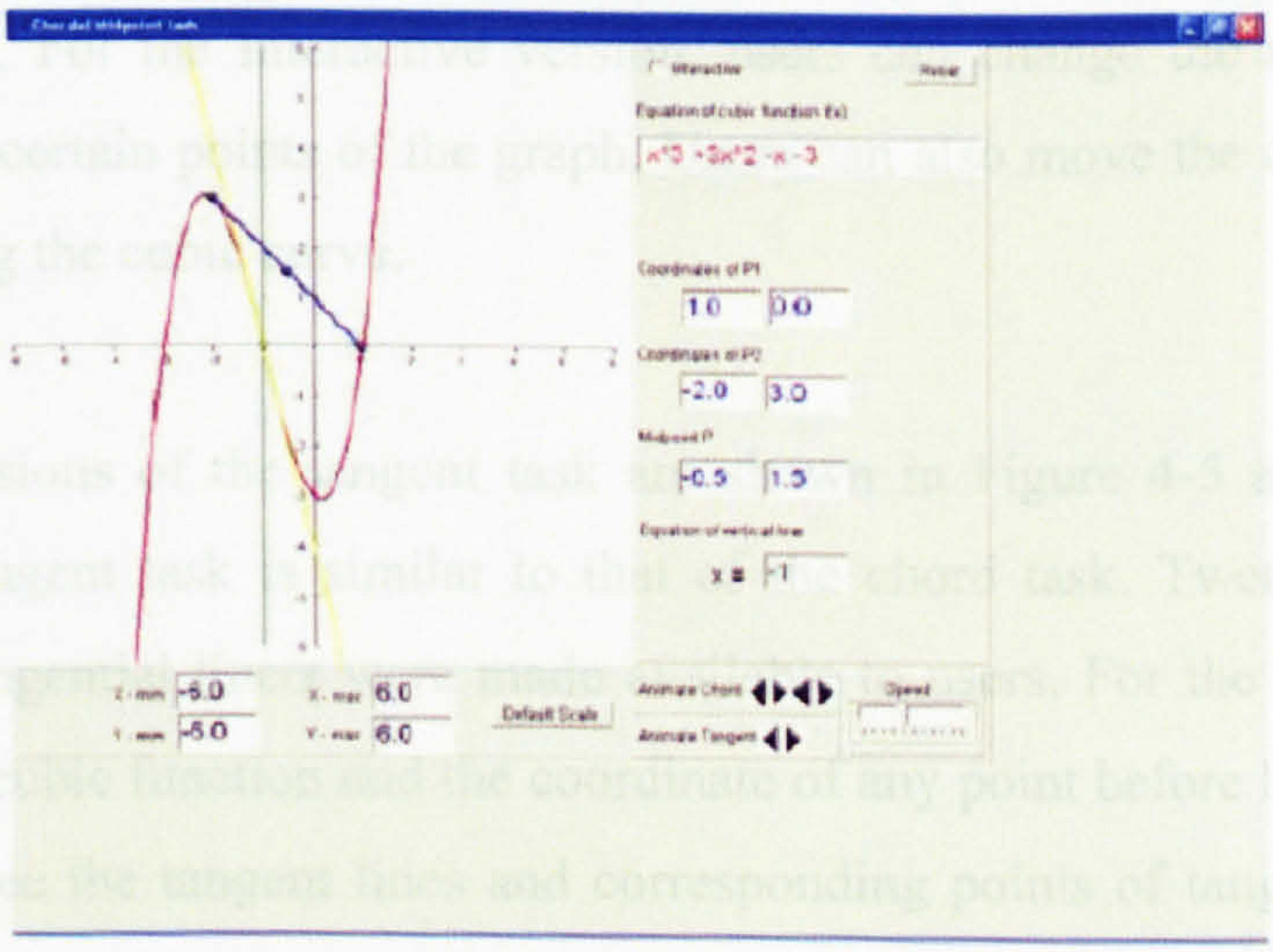


Figure 4-4 b) Dynamic Chord task (DC)

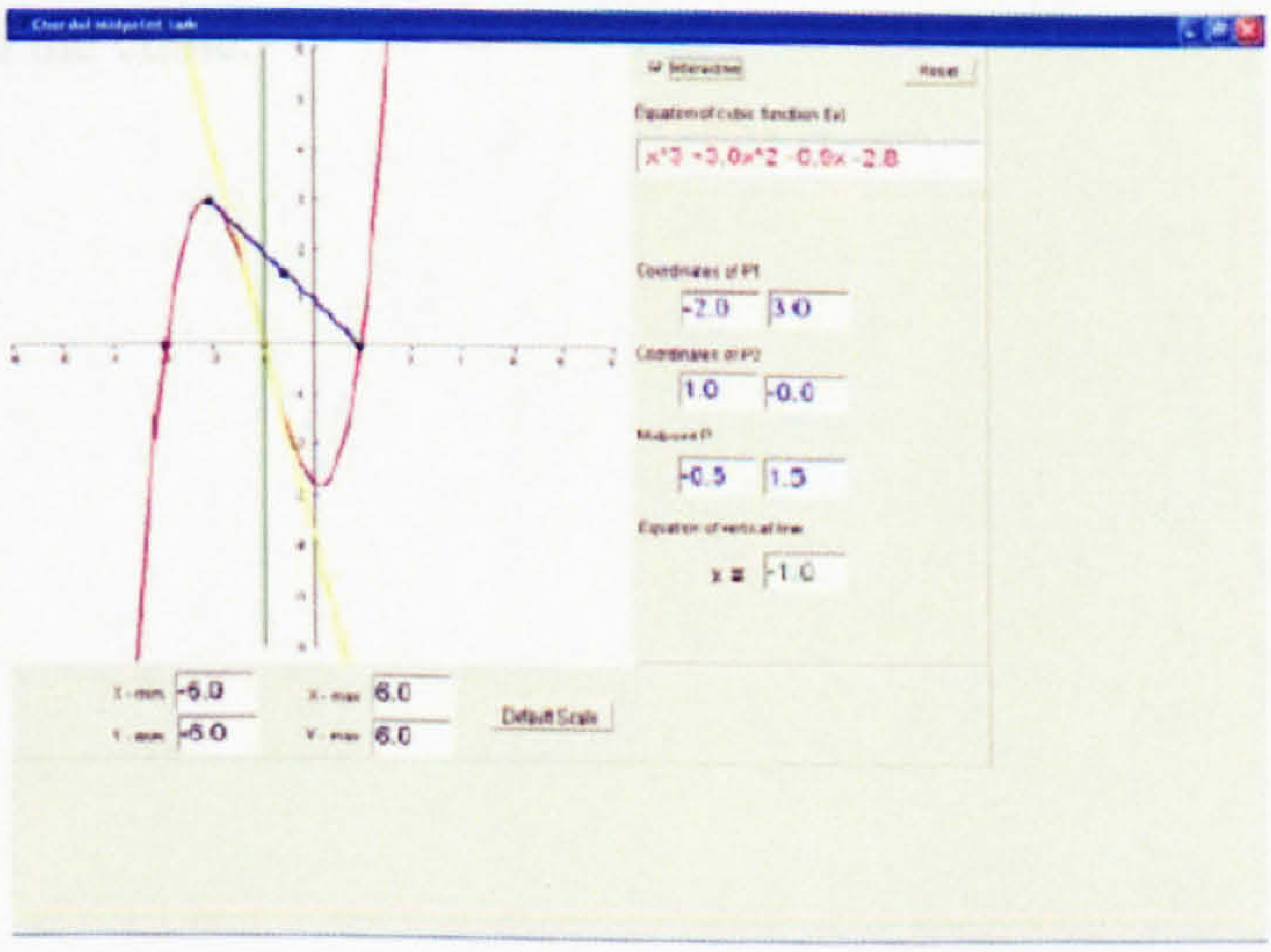


Figure 4-4 c) Interactive Chord task (IC)

Figure 4-4 Instantiations of the Chord task

Different versions of the chord task interface are shown in Figure 4-4 a, b, c. The Static version of the chord task was made similar to the Static version of the root task. Twenty pairs of slides labelled as Slide 1.a, Slide 1.b; Slide 2.a, Slide 2.b; Slide 3.a, Slide 3.b; and so on, were made available to participants. The equation for every pair remains constant, thus the cubic graph remains the same. However, for the participants to make a comparison between each pair, certain representations are different, e.g. the chord or the equation of the vertical line. For the Dynamic version of the chord task, users can enter a cubic equation and coordinates of the endpoints of a preferred chord. A midpoint is automatically computed by pressing 'enter'. The 'chord' and the 'vertical line' in the interface can then be animated along the cubic curve, to move left to right by clicking on 'animate' buttons. For the Interactive version, users can change the appearance of the graph by moving certain points of the graph. Users can also move the chord by dragging its endpoints along the cubic curve.

The different versions of the tangent task are shown in Figure 4-5 a, b, c. The Static version of the tangent task is similar to that of the chord task. Twenty pairs of cubic functions with tangential line/s were made available to users. For the Dynamic version, users can enter a cubic function and the coordinate of any point before he or she can click on 'animate' to see the tangent lines and corresponding points of tangency. By moving certain points, in the Interactive version, users can change the graph and see the changes of tangent line/s to the cubic.

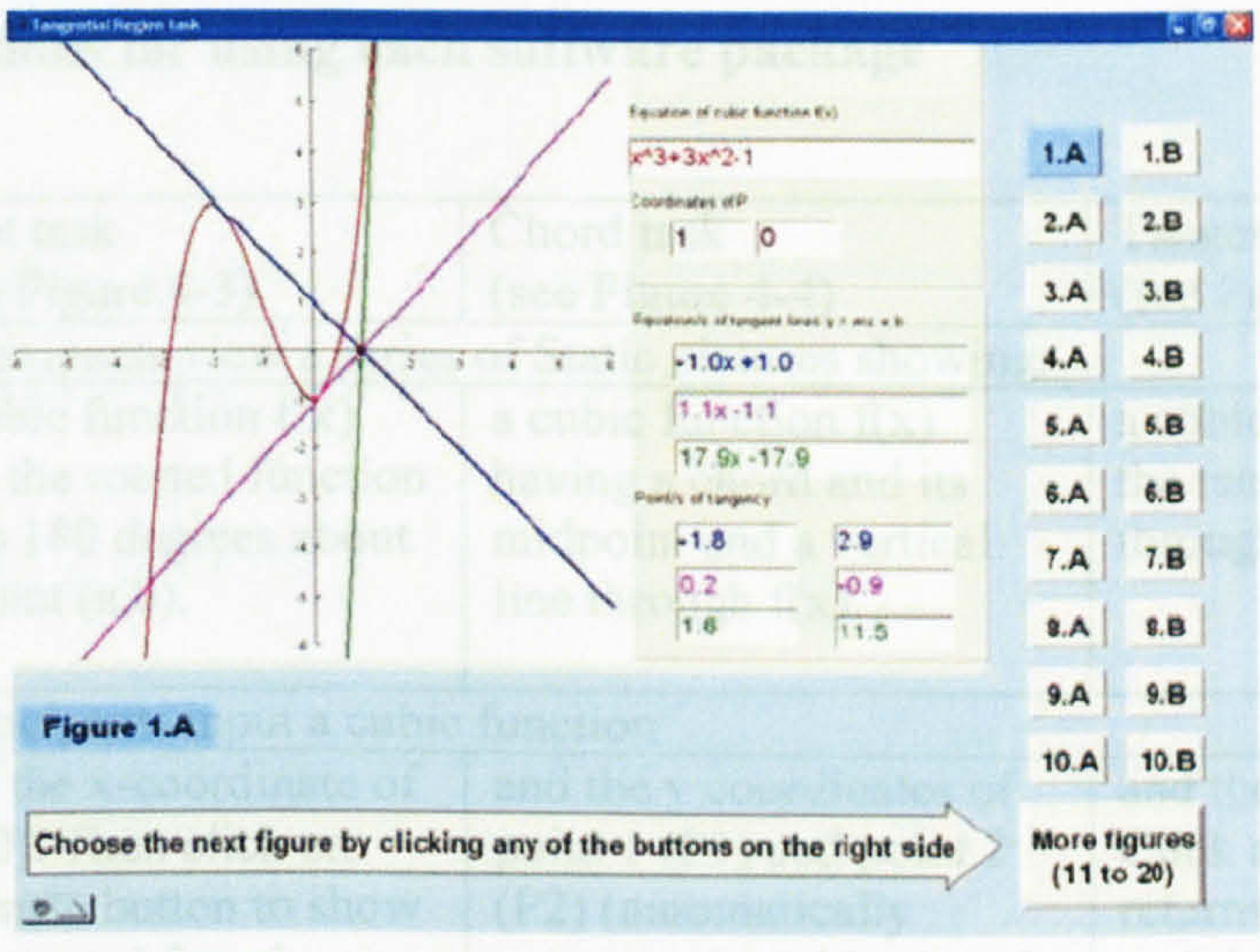


Figure 4-5 a) Static Tangent task (ST)

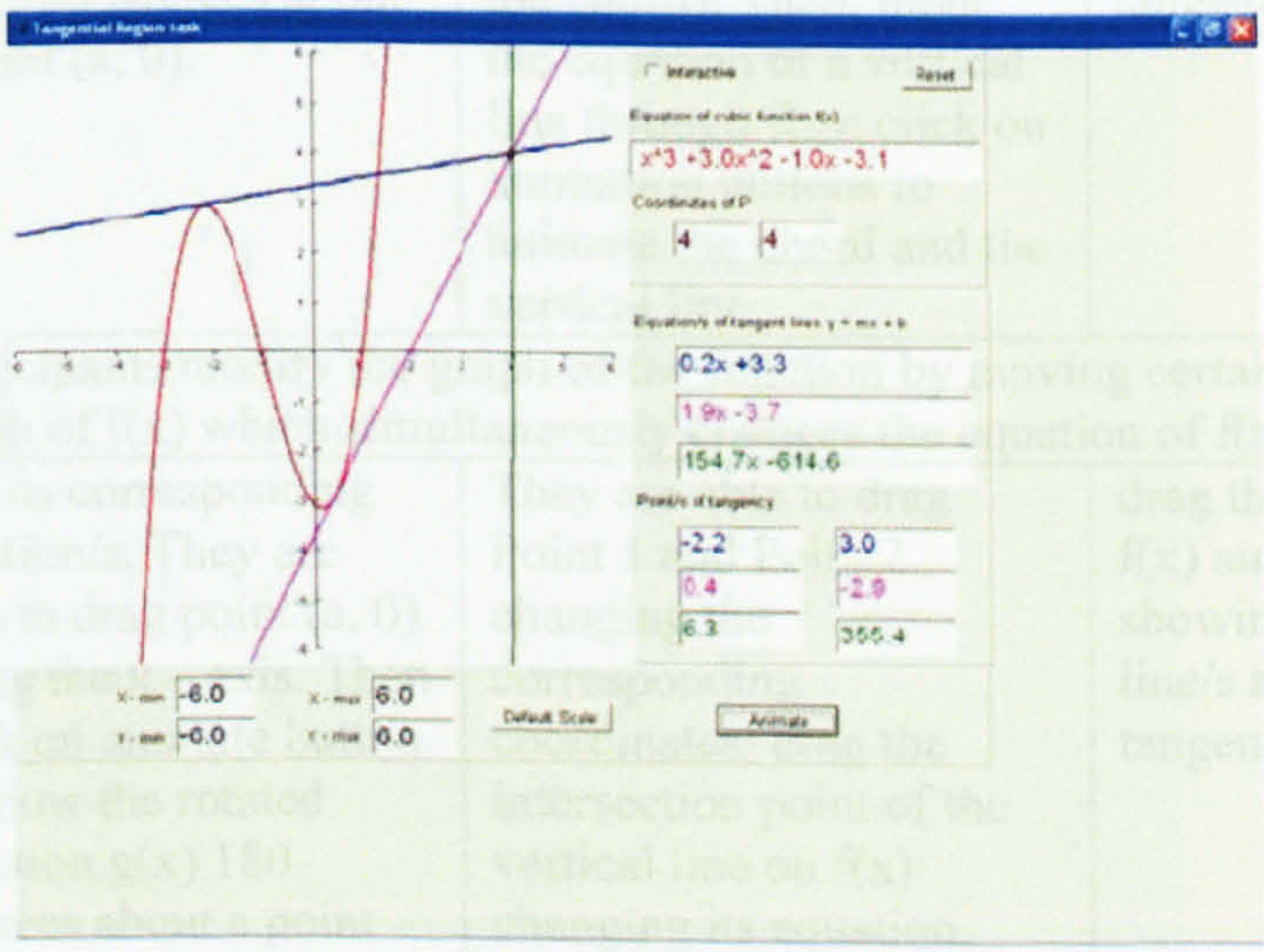


Figure 4-5 b) Dynamic Tangent task (DT)

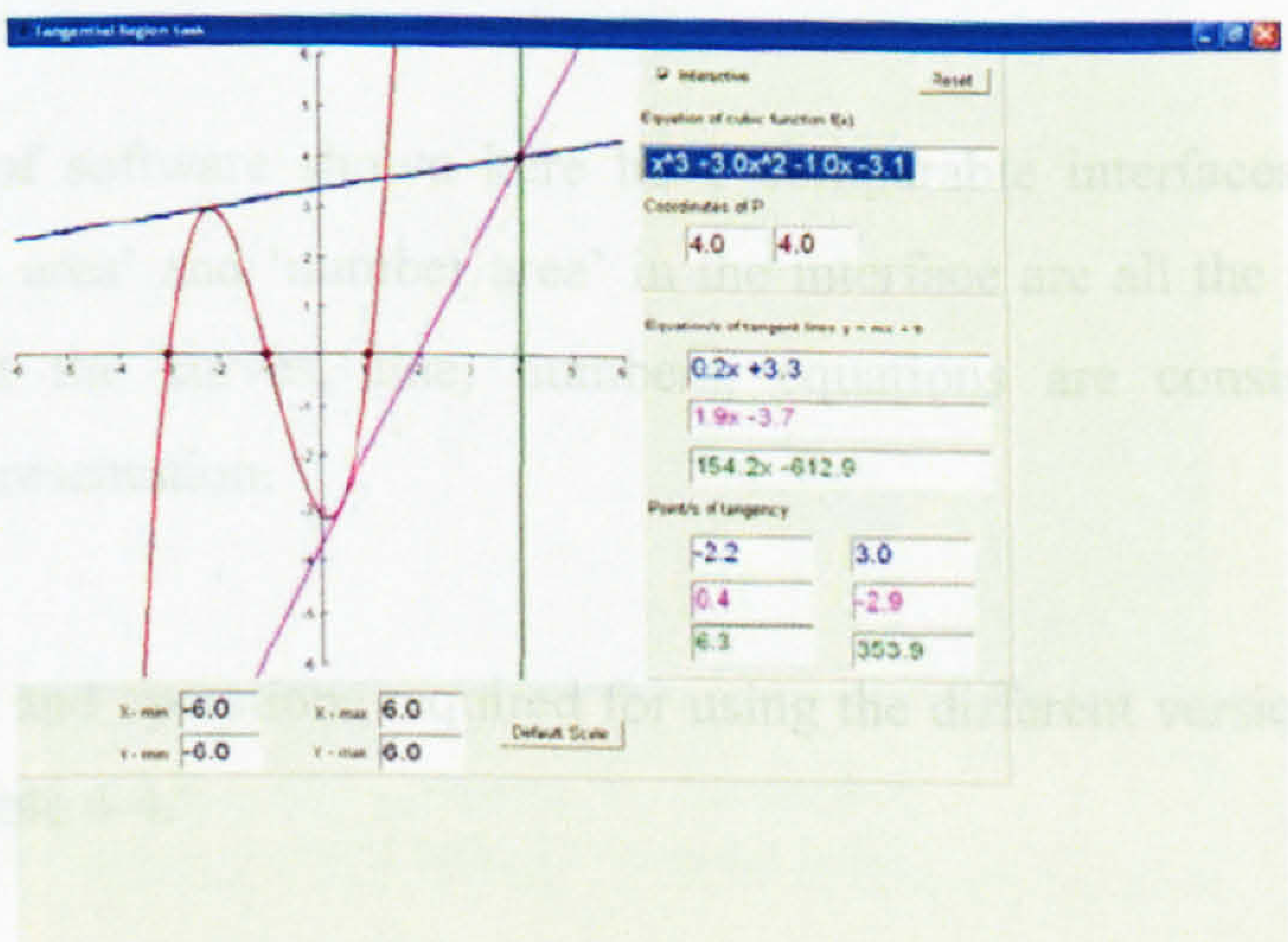


Figure 4-5 c) Interactive Tangent task (IT)

Figure 4-5 Instantiations of the Tangent task

Table 4-4 Operations for using each software package

Instantiation	Root task (see Figure 4-3)	Chord task (see Figure 4-4)	Tangent task (see Figure 4-5)
Static	Participants view a series of Static pictures showing:		
	a cubic function f(x) and the rotated function g(x) 180 degrees about a point (a,0).	a cubic function f(x) having a chord and its midpoint and a vertical line through f(x).	a cubic function f(x) and the tangent line/s through point P.
Dynamic	Participants input a cubic function		
	and the x-coordinate of (a, 0). Then click on animate button to show the rotated function g(x) 180 degrees about a point (a, 0).	and the y coordinates of point 1 (P1) and point 2 (P2) (automatically compute the midpoint of the chord); Then, input the equation of a vertical line through f(x); click on animation buttons to animate the chord and the vertical line.	and the coordinates of P; Click animate and returns the tangent line/s and the point/s of tangency.
Interactive	Participants modify the graph of the function by moving certain points on the graph of f(x) which simultaneously changes the equation of f(x)		
	and its corresponding solution/s. They are able to drag point (a, 0) along the x – axis. Then click on animate button to show the rotated function g(x) 180 degrees about a point (a, 0).	They are able to drag Point 1 and Point 2 changing the corresponding coordinates; drag the intersection point of the vertical line on f(x) changing its equation.	drag the point P not on f(x) automatically showing the tangent line/s and the point/s of tangency.

The nine pieces of software shown here have comparable interfaces. The size of the 'graph area', 'text area' and 'number area' in the interface are all the same. The colours used to represent the curves, line, numbers, equations are consistent within each corresponding representation.

The instantiations and operations required for using the different versions of the tasks are summarised in Table 4-4.

4.4 SETTING THE EXPERIMENT

This section describes the criteria upon which participants were chosen. It also includes the time and the setting and the materials used in the experiment.

4.4.1 THE PARTICIPANTS

Table 4-5 Number of participants assigned for tasks with single-instantiation and for tasks with varying-instantiations

	Code	REP	Task	Code	REP	Task
Single - instantiation	P1	S	R	P2	S	C
		S	T		S	R
		S	C		S	T
	P3	D	C	P4	D	T
		D	R		D	R
		D	T		D	C
	P5	I	R	P6	I	T
		I	T		I	R
		I	C		I	C
Varying - instantiations	P11	S	C	P16	S	T
		D	T		D	R
		I	R		I	C
	P10	S	T	P12	S	R
		I	C		I	C
		D	R		D	T
	P13	D	C	P15	D	C
		S	T		S	R
		I	R		I	T
	P8	D	R	P18	D	T
		I	C		I	R
		S	T		S	C
	P7	I	C	P14	I	R
		S	R		S	C
		D	T		D	T
	P17	I	T	P9	I	T
		D	R		D	C
		S	C		S	R
(Eighteen participants, n = 18) (Where: S – Static, D – Dynamic, I – Interactive, R – root task, C – chord task, T – tangent task)						

In accordance with the criteria relating to the design of the materials in section 4.3, the participants in this study are required to have knowledge of functions. Particularly, participants are required to have mastered graphing, identifying equation types and recognising points on the Cartesian plane. All the participants confirmed they were familiar with the prerequisite knowledge. Eighteen participants with A-level maths qualification or higher participated in the study (male = 12; female = 6; average age = 28). Two participants were assigned for each order of instantiation of the rotational design (Table 4-5). From the start of the data collection, each of the participants was assigned a code name to ensure anonymity (P1 for participant 1, P2 for participant 2, and so on).

4.4.2 MATERIALS

The materials used in the study were trialled and tested, as discussed in the next section 4.5.

Profile log. This is a record of relevant details about the participant (e.g. Name, qualifications, age, gender, date and time of experiment).

Consent forms. A form given and signed by the participants that briefly explains the experiment and its purpose, and asks for their permission to be recorded.

Checklist of materials and technical procedures. A list of the materials such as compact disc, battery, remote controller, with the sequence and order of technical operations.

Instructions notes. Nine different instructions sheets (corresponding to the task and instantiation) were created to be read to the participants.

Electronic worksheet. An electronic paper created using the Tablet PC Windows Journal which the participants were encouraged to use if they wished to write anything in order to complete the tasks.

Observation sheets. A record of the researcher's observations, used for the retrospective interview with each participant.

Technical equipment. The devices that were discussed in the previous chapter such as the DVD player, DV camcorder, TV monitor, etc.

4.4.3 TIME AND SETTING

It is recognised that in conducting qualitative studies of typical behaviour, the phenomena being investigated should, as far as possible, be observed in their natural state (e.g. Hammersley and Atkinson, 1995; Lincoln and Guba, 1985). The study, however, required a specific computer to capture what students are manipulating on the screen and what they are saying and writing whilst doing the tasks at hand. Therefore, the experiments were conducted in a 'contrived setting' (i.e. an ideal environment for special equipments (Ross and Morrison, 1996), p. 1159). The data capture set-up, as illustrated in the previous chapter, is located at the OU Institute of Educational Technology Data Capture Suite (OU IET DCS).

The experiments were first planned to be conducted in three days (one day for every task for every participant). Problems were anticipated to transpire, however, if participants are asked to come to the OU IET DCS three times. Also, since the three tasks are similar, the possibility of participants' pre-planning to complete a subsequent task could be a potential bias. It was decided to request the participant to complete all the tasks within a half day-session; in either a morning (09:00 – 13:00) or an afternoon session (12:00 – 16:00). On the average the participants of the study completed one task with the retrospective interview in 80 minutes. Time breaks were given in between sessions. Participants were given a small payment to cover expenses and time for participating.

4.5 THE RESEARCH PROCEDURES

The study was carried out according to the research plan (Figure 4-6). It describes the iterative trialling and testing of the data capture setup and the analysis system setup. It also gives the details of the pilot experiment which is a simulation of the actual experiment. It then shows the conduct of the experiment. This plan also includes the process with which participants' strategies were analysed.

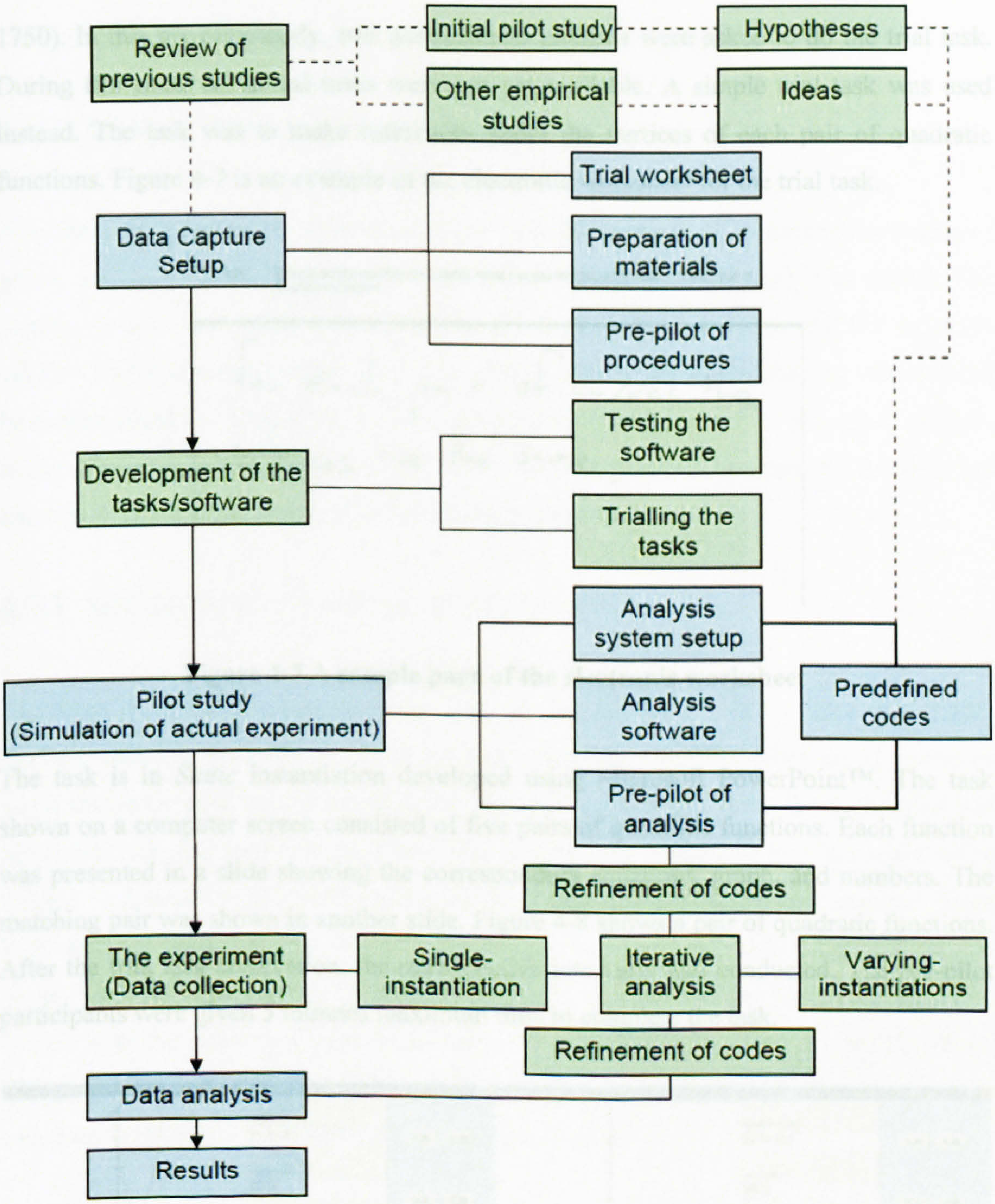


Figure 4-6 Research plan

4.5.1 CHECKING THE PROCEDURES

First, a pre-pilot study was done to double check that the data capture was working properly (this time using an upgraded latest model of the Tobii eye-tracking device ET-

1750). In this pre-pilot study, two postgraduate students were asked to do the trial task. During this time, the actual tasks were not yet available. A simple trial task was used instead. The task was to make inferences about the vertices of each pair of quadratic functions. Figure 4-7 is an example of the electronic worksheet for the trial task.

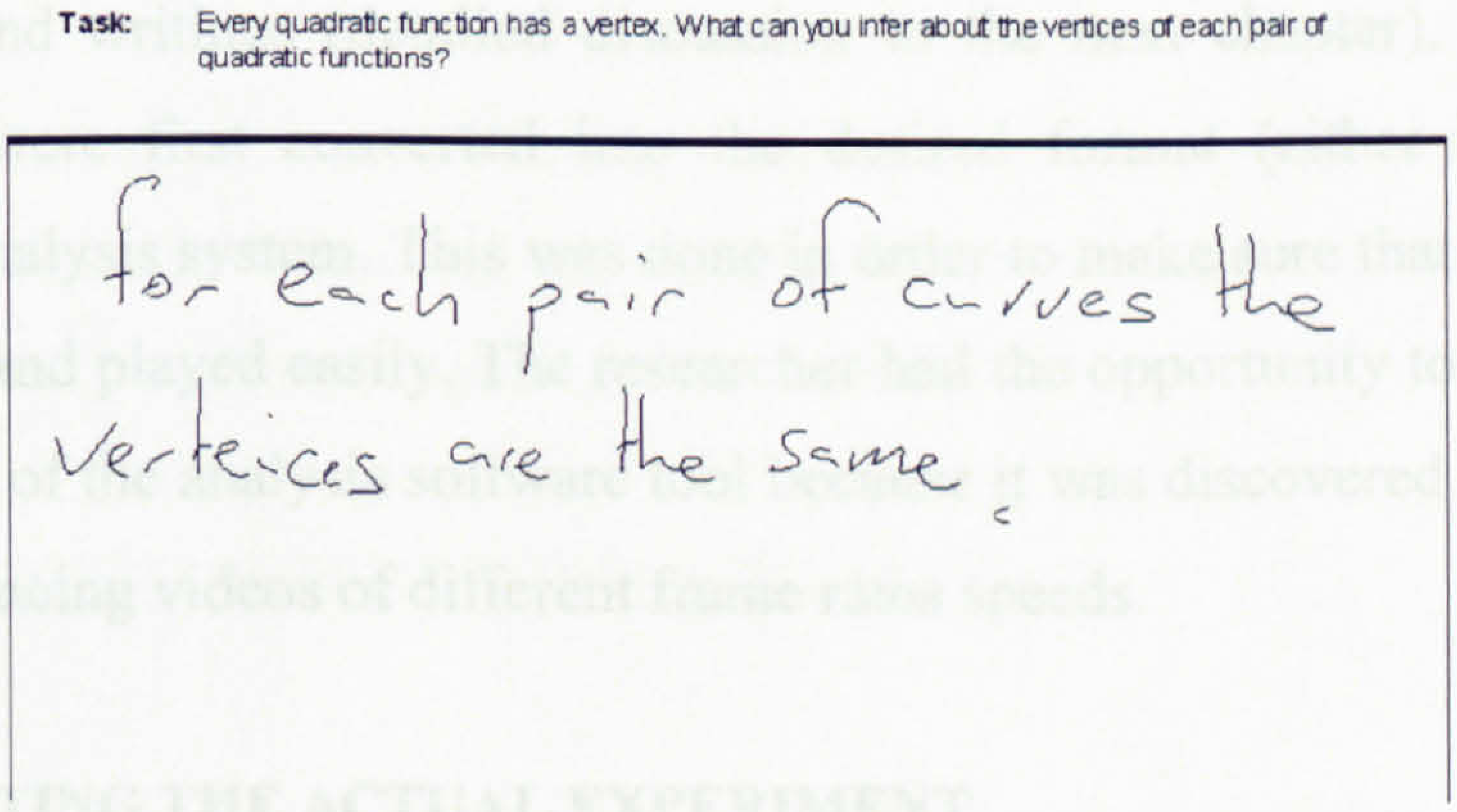


Figure 4-7 A sample page of the electronic worksheet

The task is in *Static* instantiation developed using Microsoft PowerPoint™. The task shown on a computer screen consisted of five pairs of quadratic functions. Each function was presented in a slide showing the corresponding equations, graph, and numbers. The matching pair was shown in another slide. Figure 4-8 shows a pair of quadratic functions. After the trial task completion, the retrospective interview was conducted. The pre-pilot participants were given 5 minutes maximum time to complete the task.

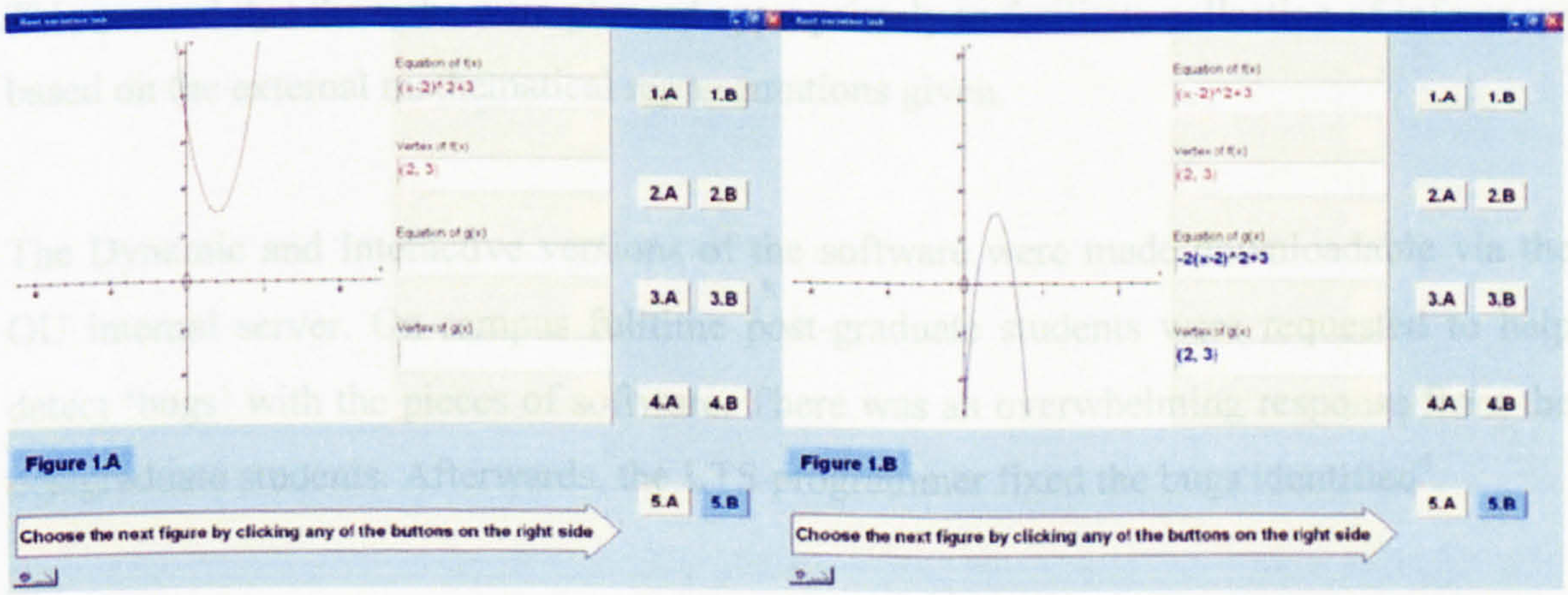


Figure 4-8 The trial task interface

This pre-pilot session ensured that the data capture was working fine.

4.5.2 Testing the Analysis System

A system, designed to coordinate and help with the analysis of simultaneous videos of gazes, actions, and writings (detailed discussion in the next chapter), was tested. The 'action videos' were first converted into the desired format (either MPEG or AVI) suitable for the analysis system. This was done in order to make sure that the videos could be synchronised and played easily. The researcher had the opportunity to suggest changes to the developers of the analysis software tool because it was discovered that the software had a problem syncing videos of different frame rates speeds.

4.5.3 Simulating the Actual Experiment

The three electronic worksheets tasks were trialled and tested with a researcher-friend. She was asked to complete the tasks without recording. The researcher asked questions when the participant was observed to be getting confused and when the participant had difficulty with the instructions given. This enabled the procedure and materials to be refined. Afterwards, six mathematics students and three mathematics teachers from University of Santo Tomas, Manila, Philippines answered the task using 'pencil and paper.' The inferences given by the students and teachers from the University of Santo Tomas were the inferences expected to come up when conducted in the United Kingdom. This assured that the tasks were phrased appropriately to facilitate collection of inferences based on the external mathematical representations given.

The Dynamic and Interactive versions of the software were made downloadable via the OU internal server. On-campus fulltime post-graduate students were requested to help detect 'bugs' with the pieces of software. There was an overwhelming response from the postgraduate students. Afterwards, the LTS programmer fixed the bugs identified[4].

[4] There was still a minimal bug found during the conduct of the experiment related to the 'scale zoom function'. This did not affect the data that were collected

After having tested and trialled the software and the worksheets, a pilot session simulating the real experiment was conducted. This involved 3 tasks plus a trial task with retrospective interviews. Four postgraduate students with A-level Maths qualifications or higher participated in the pilot study. Improvements were made after each participant. The final materials were used for the actual experiment.

The data from this pilot experiment were manipulated and pre-analysed to further improve the procedure for the actual experiment. The next section describes the resulting procedure based on these iterative tests.

4.6 THE EXPERIMENTAL PROCEDURE

After conducting all the pre-experimental studies, a detailed procedure was, as much as possible, strictly and consistently followed in the actual experiments. The data collection consisted of three phases (Figure 4-9).

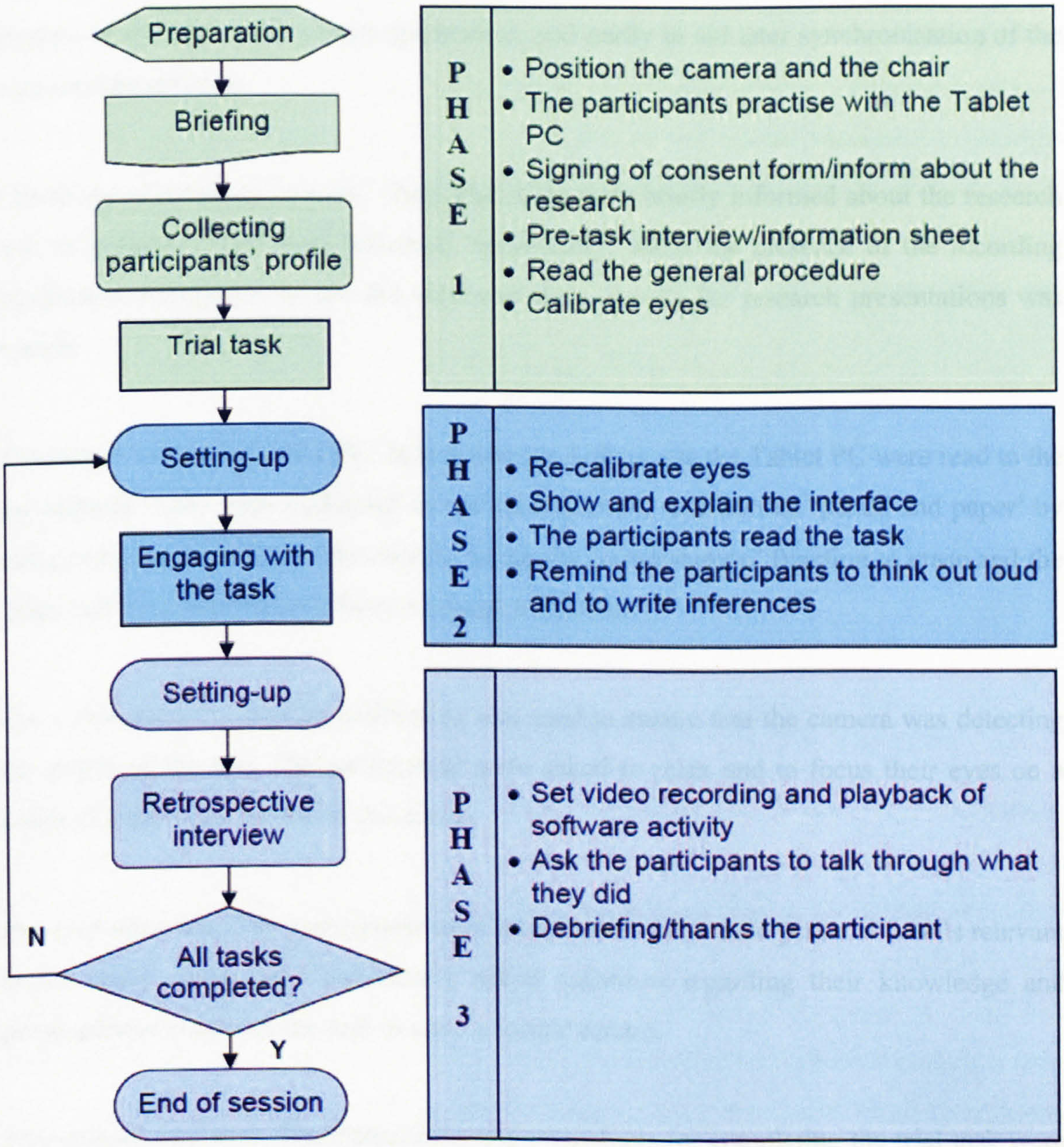


Figure 4-9 Data collection procedure

4.6.1 PHASE 1 OF DATA COLLECTION

The first phase involved preliminaries such as the obtaining of informed consent; practice in using the Tablet PC; eye calibration; a pre-task interview to elicit background information relevant to the study; and a five-minute trial task to help make participants comfortable with the setup and procedures. All this was recorded, partly to check all aspects of the data capture were operational, and partly to aid later synchronisation of the various data streams.

Obtaining of informed consent. The participants were briefly informed about the research and its purpose. They were informed, specifically, about the presence of the recording equipment. Permission to use the videos of their activity for research presentations was sought.

Practice in using the Tablet PC. Instructions on how to use the Tablet PC were read to the participants. They were instructed to use the Tablet PC as a normal 'pencil and paper' by using only the 'pen-button' function to write; the 'eraser-button' function to erase and the 'page-tab' function to move from one page to another.

Eye calibration. A 16-point calibration was used to ensure that the camera was detecting the pupils of the eye. The participants were asked to relax and to focus their eyes on a series of calibration points on the screen.

Pre-task interview. The participants were interviewed about their personal details relevant to the study. They were particularly asked questions regarding their knowledge and preparedness to answer the task to have a formal record.

Five-minute trial task. The instructions and procedures for completing the trial task were the same as that of the actual task discussed next in Phase 2.

4.6.2 PHASE 2 OF DATA COLLECTION

The second phase of data collection consisted of the task. Participants were asked to think aloud (Ericsson and Simon, 1984), to help clarify what thought processes might be occurring during gazes. Scanlon (1993) has noted that asking an individual person to think aloud is not always the best way to collect talk data. Participants deeply involved in a task can sometimes fail to talk. Researchers can capture more utterances from discussions made between a pair of participants than a single participant. Although pairs can be more effective, the particular data capture setup allowed only one person's eyes to be tracked, and it was decided for this study to focus on individuals' responses to multiple representations. Once this data capture setup has proved itself, further research could examine the benefits of participant pairs. It was therefore important for the researcher to remind participants to keep talking out loud, as suggested by van Someren (1994) and Erickson and Simon (1984).

Task procedures. The procedures followed in order were 1) reading of the instructions 2) reading of the task 3) reminding of the time limit 4) asking non-prompting questions such as *"What are you doing now?" or "Please talk aloud."* The detailed procedure and instructions given to participants are presented in Appendix A.

4.6.3 PHASE 3 OF DATA COLLECTION

Although lack of talk is not completely disastrous here because the researcher still has the evidence of gazes and written notes, immediately after the task participants were shown a video replay of what they did, with their eye gazes superimposed on the screen, and asked about their reasons for focusing on particular representations. While recognising that there are questions about the extent to which participants' engage in post hoc rationalisations on the basis of such gaze records, the kind of retrospective interview used in this third phase of data collection was intended to help participants recall their action and thought processing (Eger, 2005; Hansen, 1991; Hansen et al., 2001), not just to elicit talk data that the participant may have failed to say but also to validate interpretations of what the participant had said during the activity. Doing retrospective reporting may help to verify pre and post interviews as suggested by Lewalter (2003) recognising that retrospective interviews can get a closer insight on the cognitive process.

One participant chose not to work on the task by looking at the computer screen, but preferred to work almost exclusively on the Tablet PC. In such cases, the recording of the writing and sketching on the Tablet PC – instead of the eye-tracking recording – can serve as the context for the retrospective interview. This flexibility is clearly a strength of the date capture setup.

Retrospective interview procedure. The 'gaze video' of a particular task was shown to the participants. The 'gaze plots' have been found to serve as visual cues that can aid participants in recalling what they did when interacting with computers (see e.g. Eger, 2005). They were asked to talk the researcher through what they did, what representations they were considering and why, the sort of inferences they were deriving from start to finish, the reasons for these inferences, and the reasons for changing inferences, if any, and why. They were then asked about their experience with the task, particularly in relation to the task, instantiation, interface, and representations, and if they had felt any discomfort whilst doing it. Examples of questions asked were: *what were you doing here? Could you tell me why you were looking at these...?* In instances where a participant mostly used the Tablet PC, the 'writing video' was shown instead of the 'gaze video.'

4.7 SUMMARY

This chapter has presented the details of the use of the data capture system to collect learners' interaction with mathematical representations instantiated in Static, Dynamic and Interactive forms. It also has given the justifications for the rotational design adopted, the development of the tasks and the software, the procedures followed, and the set-up of the experiment. This chapter has, moreover, established some of the strengths of capturing learners' gazes, utterances, writings and actions with multiple mathematical representations and has described the measures undertaken to reduce possible problems arising in data collection.

Approaches to data analysis are considered in detail in the next chapter.

5 APPROACHES TO ANALYSING LEARNERS' STRATEGIES WITH MULTIPLE REPRESENTATIONS

5.1 INTRODUCTION

The aim of this thesis is to identify how representations instantiated in different ways influence learners' cognitive processes. The investigation of strategies that learners use in terms of the representations offered to them has hinted at some explanations as to why learners change their strategies (San Diego, 2003). Approaches to analysing video data of learners' computer interaction with multiple representations in terms of strategies are discussed in this chapter.

When learners are asked to verbalise their thoughts about multiple mathematical representations, some researchers can be left in the situation of analysing utterances based on video records of activity which may have ambiguous signifiers. They are also faced with post hoc analysis of paper-based worksheets, in which temporal order has to be guessed. Researchers have also faced profound difficulties in coordinating video data of learners and screens with the paper-based data of what learners write, with the field-notes that indicate where learners' attention might be directed, and with the transcriptions of what learners say. Previous studies have tried different ways to face these challenges. Some research has used a variety of analysis tools such as Timelines (see e.g. Issroff, Scanlon, and Jones, 1997) and Transana™ (Blake and Scanlon, 2002; Scanlon, Blake, Joiner, and O'Shea, 2005). Many software packages are available to help researchers manage and coordinate video recordings; from video editing technologies down to qualitative analysis tools. For example, there is low cost software for combining video streams and editing (Spiers, 2004), and freeware analysis tools such as the one developed at the University of Wisconsin-Madison (Woods, 2005). In using video analysis tools, Knoll and Stigler (1999) emphasise the value of features such as category indexing, annotation for every event, and support for processing of data such as graphical representations of coding hierarchies.

This chapter begins with a description of how the latest analysis software enables multiple video feeds, eye gazes, handwriting, and verbal transcripts to be synchronised and coded. It explains how these techniques are being used to help identify the strategies more accurately than has been possible with previous data collection technology. It then discusses the different forms of analysis that can be derived from the data. The phases of analysis are described in detail.

5.2 COORDINATING VIDEOS, ANNOTATIONS, CODINGS AND TRANSCRIPTIONS

As discussed earlier in section 3.2.2, analysing videos can be time consuming. Although there are now analysis tools that can search for a *code* and link it to a particular *event* in a video, only a few tools are capable of coordinating multiple video streams. Moreover, the majority of video analysis tools that are commercially available are for analysing action patterns (e.g. Sportscode™ for detecting patterns across athletes' performance); and do not allow for coordinating detailed transcription. After thorough investigation of available software tools, the tool that was identified as most suitable for help in coordinating simultaneous videos streams, codings, and transcriptions was identified. In addition, the software company agreed to tailor the tool to allow coordination of gaze video and eye-tracking data. This feature takes the burden of coding a 'gaze video'.

The software tool responsible for generating eye-tracking data is first discussed. This is followed by an outline of the video analysis software tool that can import this eye-tracking data.

5.2.1 EYE-TRACKING SOFTWARE

The software used for capturing eye gazes, ClearView™ (Tobii, 2003), also allows for quantifying gaze data by defining Areas Of Interest (AOI) within the interface. The software generates log time stamps and fixation durations on those AOI as the activity progresses. The resulting log file can provide statistical information on participants' attention on those AOI. The log file can be imported as a text file that can be fed to a compatible software tool for qualitatively analysing gaze data. ClearView also generates an "Event data" file, consisting of timestamps of participants' mouse clicks and keyboard operations.

ClearView generates the gaze video in a readily analysable format (i.e. AVI file) that can be played using state-of-the-art video analysis software compatible with the eye-tracking software. The challenge of coordinating and analysing the multiple streams of data – gaze, actions, event log, transcript, writing, and retrospective utterances is considered next.

5.2.2 VIDEO ANALYSIS SOFTWARE

Following Candy, Bilda, Maher and Gero (2004), and after careful evaluation of available tools, the software package INTERACT™ (Mangold Software and Consulting GmbH, 2005) was selected. The software can import the eye-tracking and software data text files generated by ClearView, which makes it easy to watch eye gazes and software event logs. Moreover, INTERACT can help coordinate transcription, annotation and coding of video data with the eye-tracking data. Unlike many other video analysis software packages, INTERACT is capable of analysing multiple video streams simultaneously. There is no need for an editing system because the software can save an offset time for syncing multiple video streams. Transcription and detailed description of a coded event can be synchronised using INTERACT. Video streams are played in separate floating windows. This is advantageous because a researcher can manipulate the size of the screens. Moreover, if working with a double monitor setup, one can drag all the video windows across a second monitor (see Figure 5-1). This allows researchers to see synchronised video episodes by a simple mouse click.

INTERACT's interface is similar to that of a spreadsheet (see Figure 5-1). A defined category (e.g. action, writing, AOI, software event, and so on) appears as a column name. The rows then consist of events. Codes go in the intersections between a category and an event and can include a detailed description. Events can be organised into an "Episode" –

a small collection of sequential coded events – and they can also be sorted and searched by means of the codes and categories.

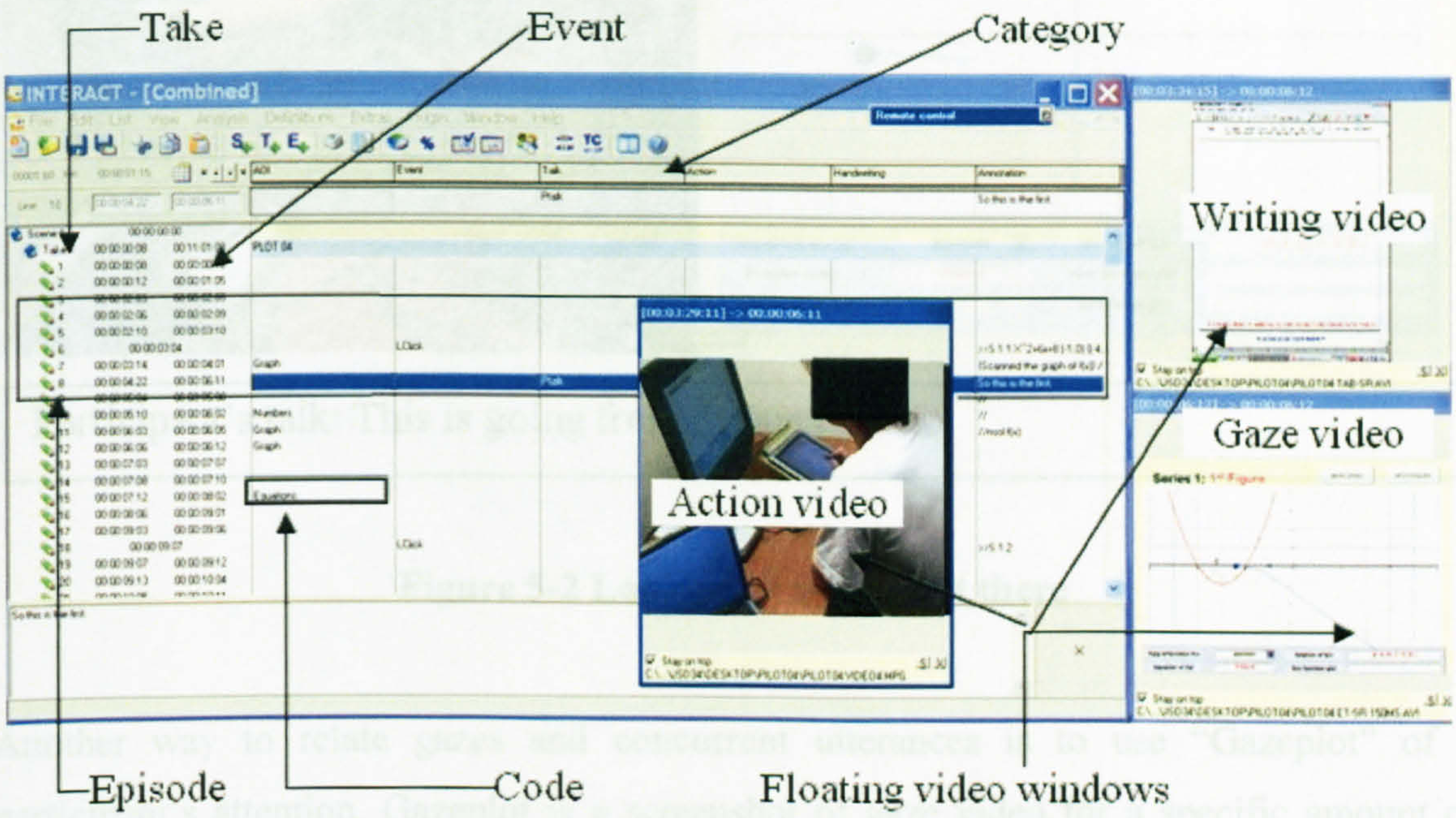


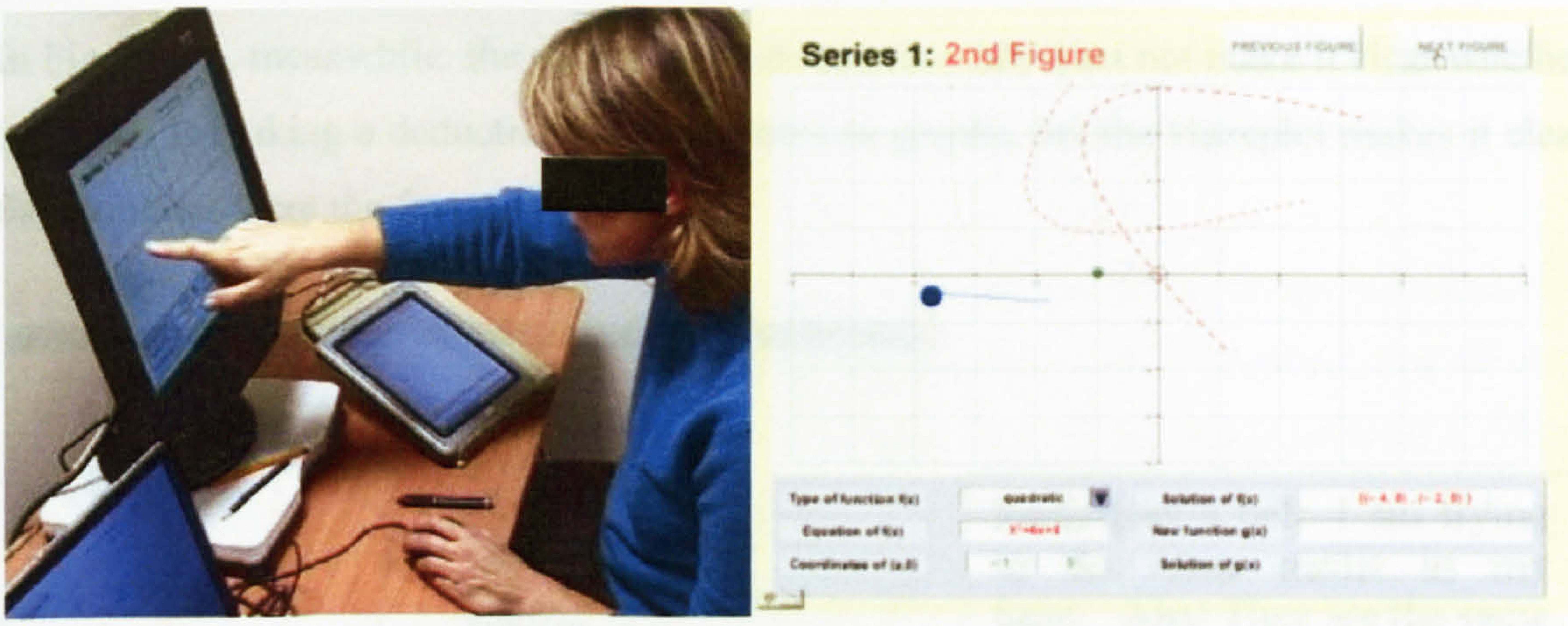
Figure 5-1 Use of INTERACT to coordinate multiple video streams

5.3 INTEGRATED ANALYSIS OF GAZES, ACTIONS, WRITING AND UTTERANCES

The study of learners' dynamic computer interaction entails examining the gaze video of participants' activities directly in combination with utterances, notes, and sketches. Analysing these sets of behaviours can provide a more complete picture of learners' engagement with computer based representations.

5.3.1 RELATING THE GAZE VIDEO AND THE ACTION VIDEO

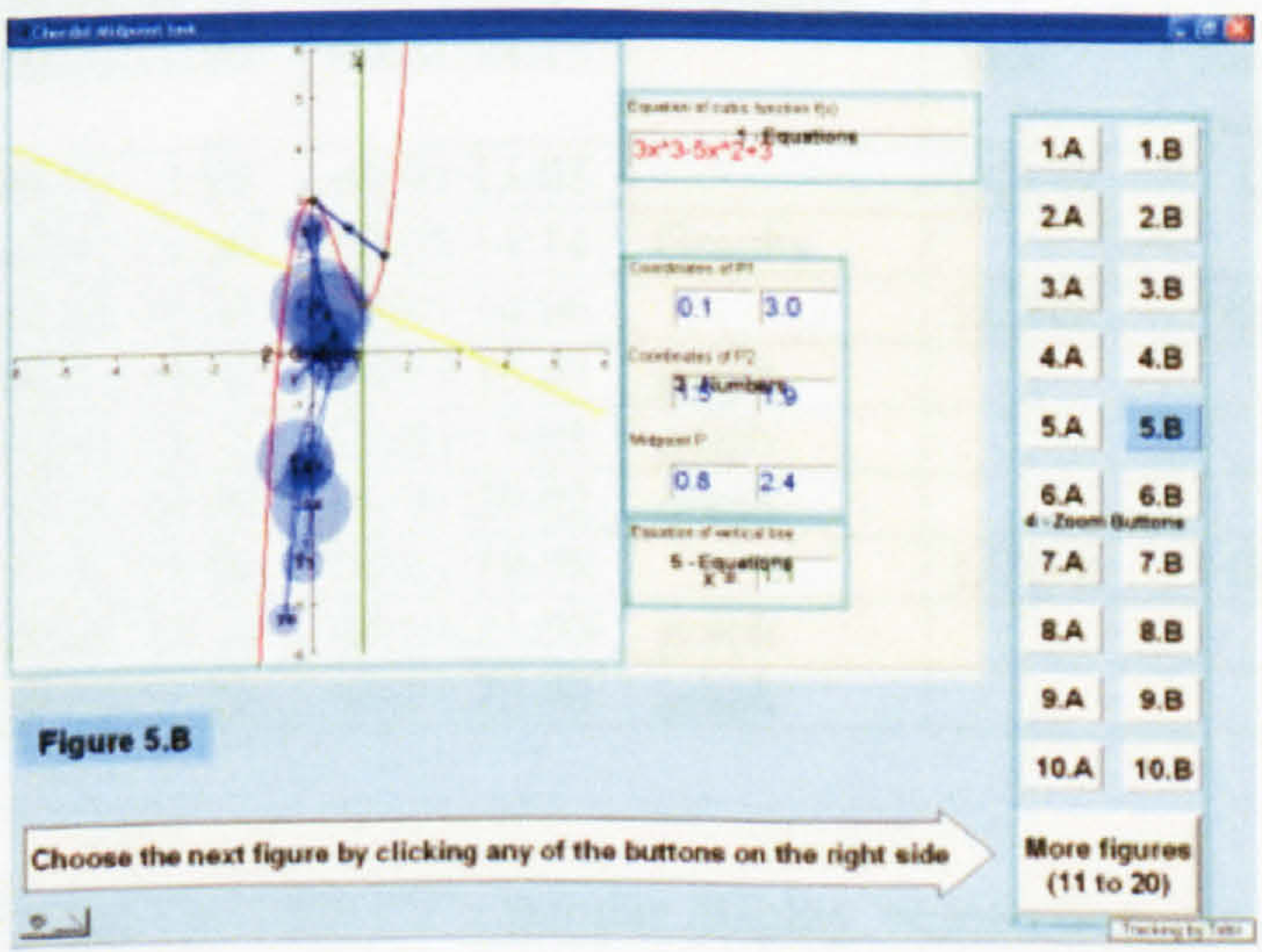
There are several advantages to being able to watch gaze with action simultaneously. One important one allows for analysing 'unseen events.' such as when learners imagine something using their eyes. For example, in an episode taken from the study (see Figure 5-2), the gesture, spoken words and gaze video, combine to create a strong sense that the participant is visually imagining a moving point.



Participant's talk: This is going from minus two...

Figure 5-2 Looking at what isn't there

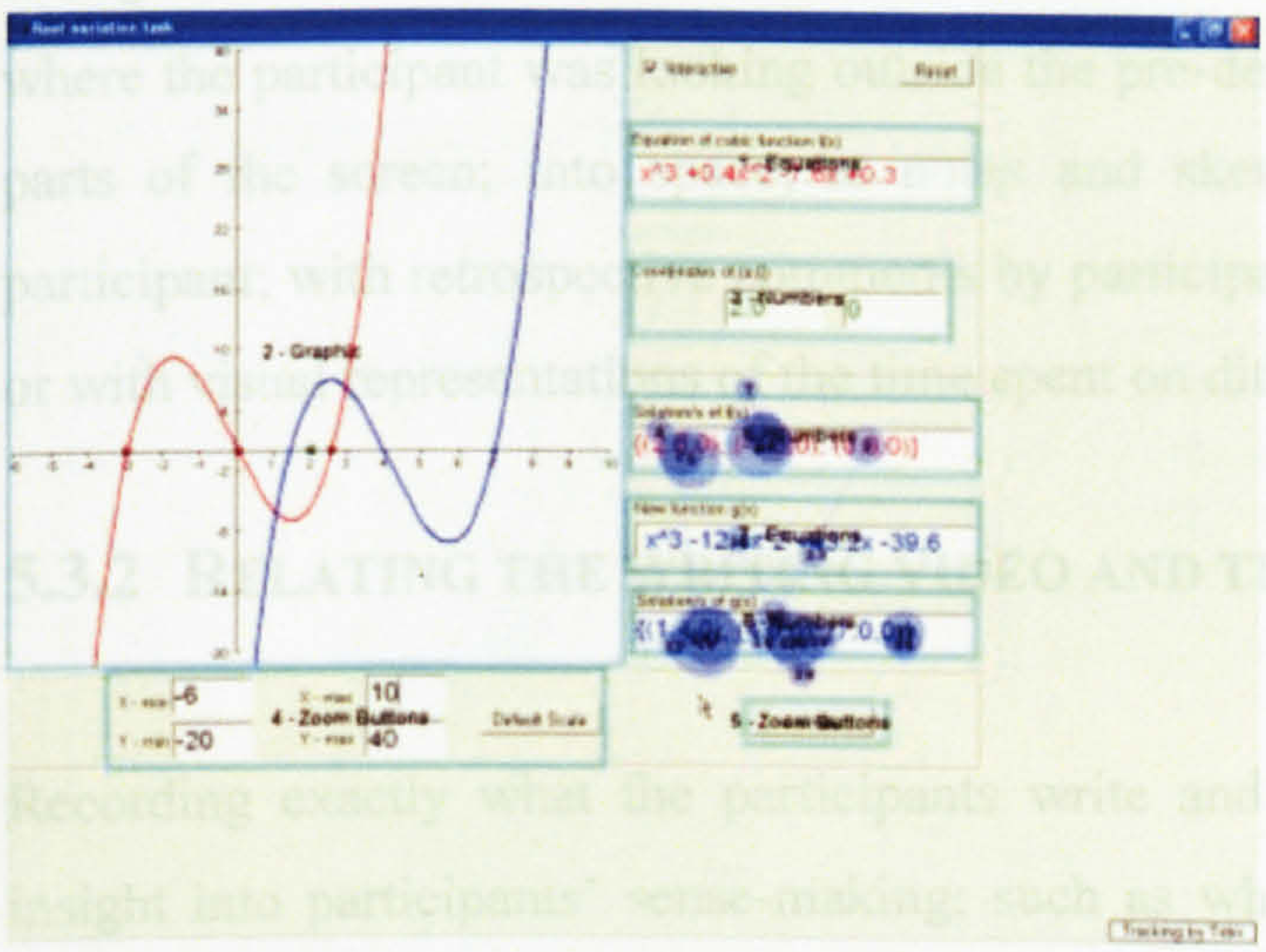
Another way to relate gazes and concurrent utterances is to use "Gazeplot" of a participant's attention. Gazeplot is a screenshot of gaze video for a specific amount of time. Using ClearView, screenshots of what participants are seeing as utterances are made. By analysing Gazeplot, for example, Figure 5-3 shows a situation in which the participant appears to be imagining a line with their eyes. There is much evidence in the data collected of participants visually tracing graph lines, and predicting the behaviour of representations on empty regions of the screen.



Participant's talk: I'm trying to imagine what happens as the line tends to infinity...where will the points lie... Err... which area?

Figure 5-3 Gazeplot showing imagining of a line

In Figure 5-4, meanwhile, the participant's concurrent talk does not make it clear whether he or she is making a deduction from numbers or graphs, but the Gazeplot makes it clear that numbers were the focus of attention.



Participant's talk: I am trying to do some maths in my head…Aha! They are the same distance away.

Figure 5-4 Gazeplot revealing the basis for a deduction

Table 5-1 Episode showing integrated gaze, actions and transcript

Time code		See	Do	Utterance/Description of action
Entry	Exit	Reps	Event	Annotation
00:01:00:08	00:01:00:13	Graphs		((Interacting with slide 3))
00:01:02:11	00:01:04:12	Numbers		
00:01:06:03	00:01:06:09	Numbers		
00:01:07:12	00:01:09:05	Numbers		
00:01:12:03	00:01:14:15		Talk	Err... I'll go back. Just because it is easier ((laughed))
00:01:13:05	00:01:13:05		LClick	< Goes back to slide 2
00:01:14:00	00:01:14:14	Graphs		
00:01:14:06	00:01:14:06		LClick	<Goes back to slide 1
00:01:15:00	00:01:15:07	graph		
00:01:16:12	00:01:17:01	graph		
00:01:18:09	00:01:20:02	graph		
00:01:19:08	00:01:19:08		LClick	>Goes forward slide 3
00:01:20:10	00:01:21:00	graph		
00:01:21:10	00:01:22:00	graph		

Using INTERACT's tabular display, researchers can see critical shifts of attention. What a learner is looking at can be generated automatically using ClearView and can then be imported onto INTERACT. Table 5-1, an episode taken from this study, shows how integrating gaze, actions and transcript can provide the context for a shift in attention from numbers to graphs that utterances in isolation would fail to illuminate.

It might be desirable in some studies to enhance this augmented transcript with details of where the participant was looking outside the pre-defined Areas Of Interest (e.g. at other parts of the screen; into space; at notes and sketches; at the keyboard; at a fellow participant; with retrospective comments by participants; with the researcher's fieldnotes; or with visual representations of the time spent on different actions, gazes or utterances.

5.3.2 RELATING THE WRITING VIDEO AND THE ACTION VIDEO

Recording exactly what the participants write and sketch in real time provides some insight into participants' sense-making; such as when they return to look at something they have written previously, change what they have written or employ representations not presented to them. Figure 5-5 shows two different ways in which participants recorded information.

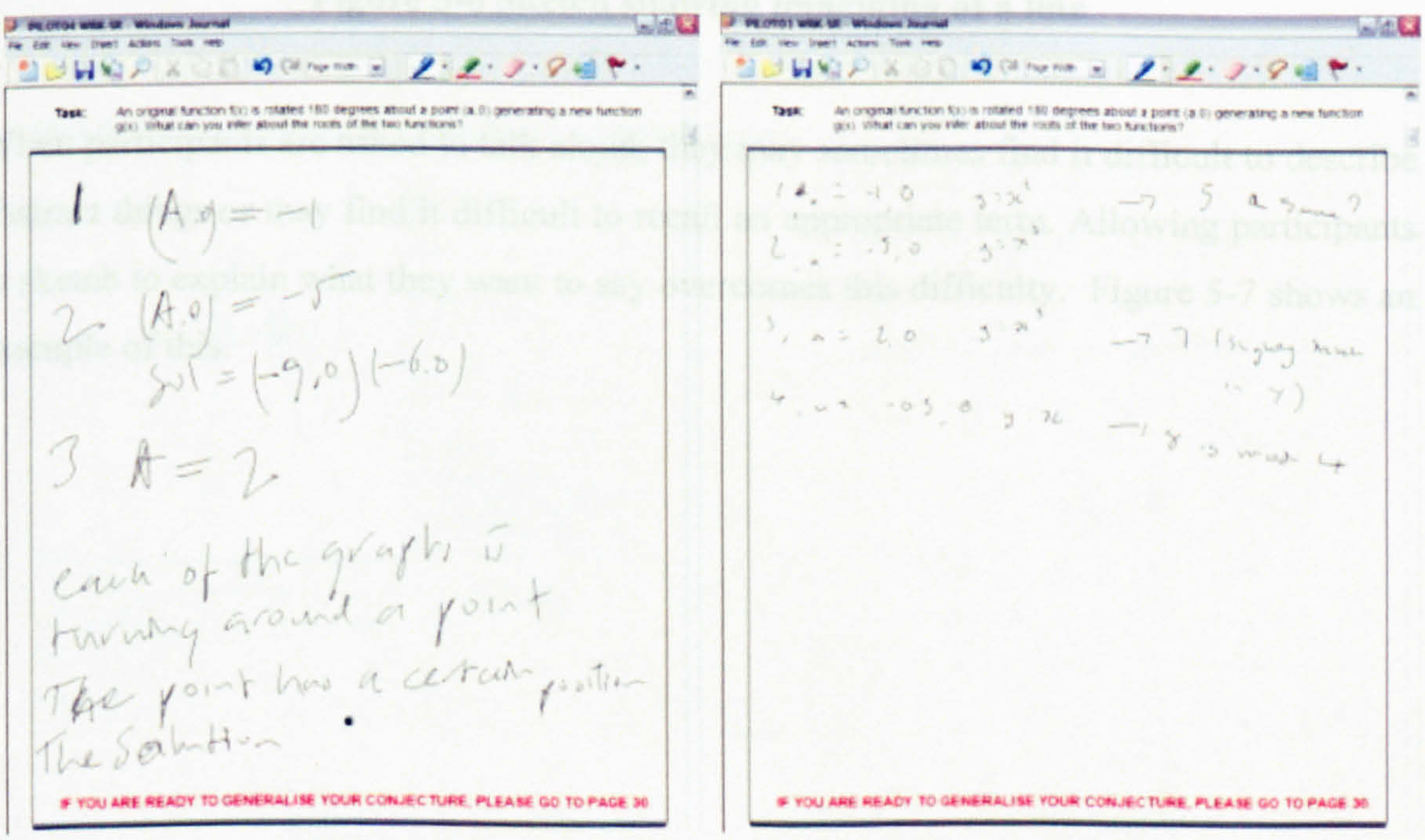
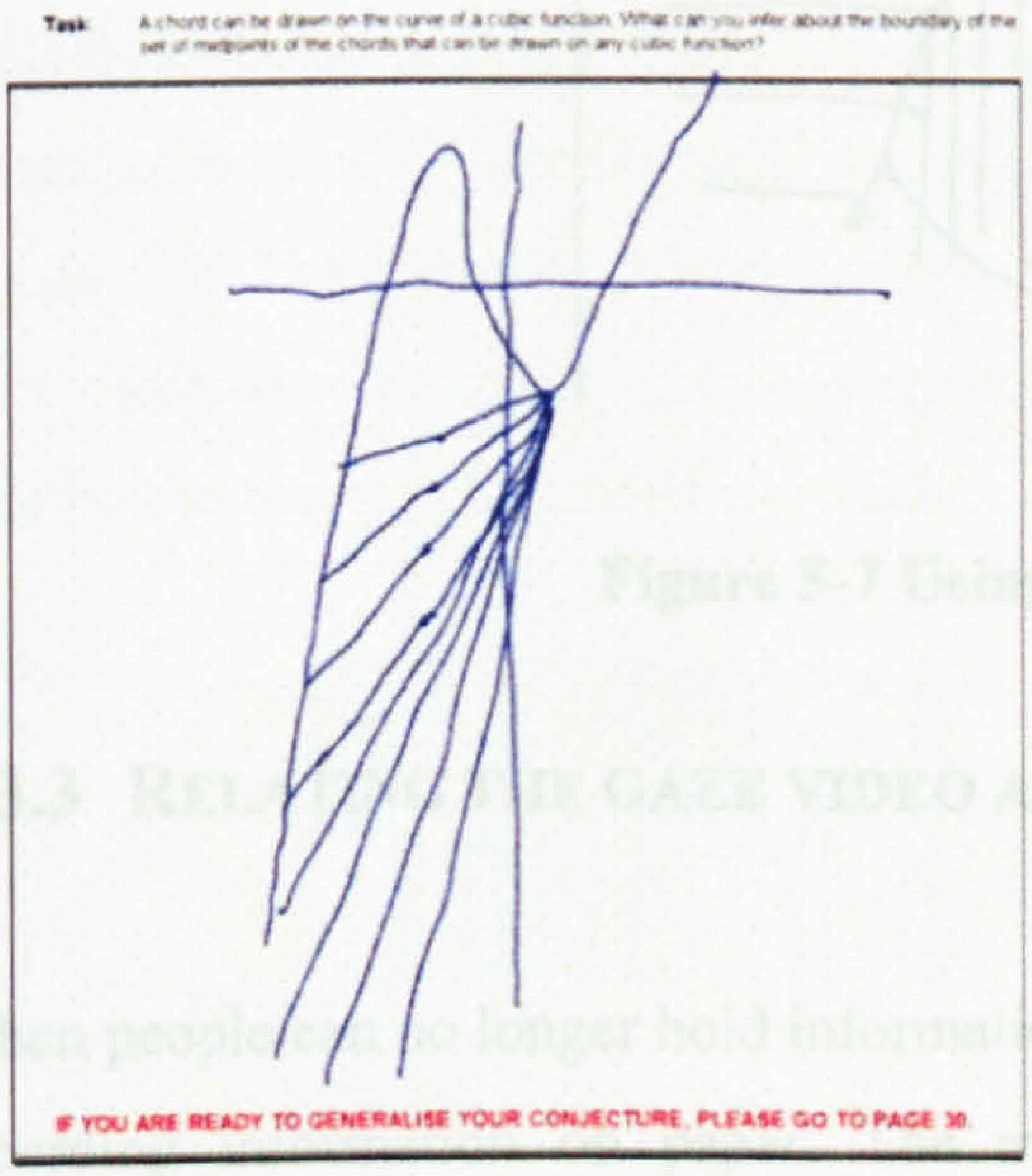


Figure 5-5 Two written responses

Some participants abandon working on the screen and think by sketching (Villarreal, 2000). Combining a Tablet PC with analysis software such as INTERACT can highlight instances in which participants make sketches (Figure 5-6) to help them imagine, instead of imagining using their eyes (compare with Figure 5-2).



Participant's talk: I'm trying to imagine… this line as it goes to the extreme… what will happen?

Figure 5-6 Sketch showing imagining of a line

When participants are asked to talk aloud, they may sometimes find it difficult to describe abstract things or may find it difficult to recall an appropriate term. Allowing participants to sketch to explain what they want to say overcomes this difficulty. Figure 5-7 shows an example of this.

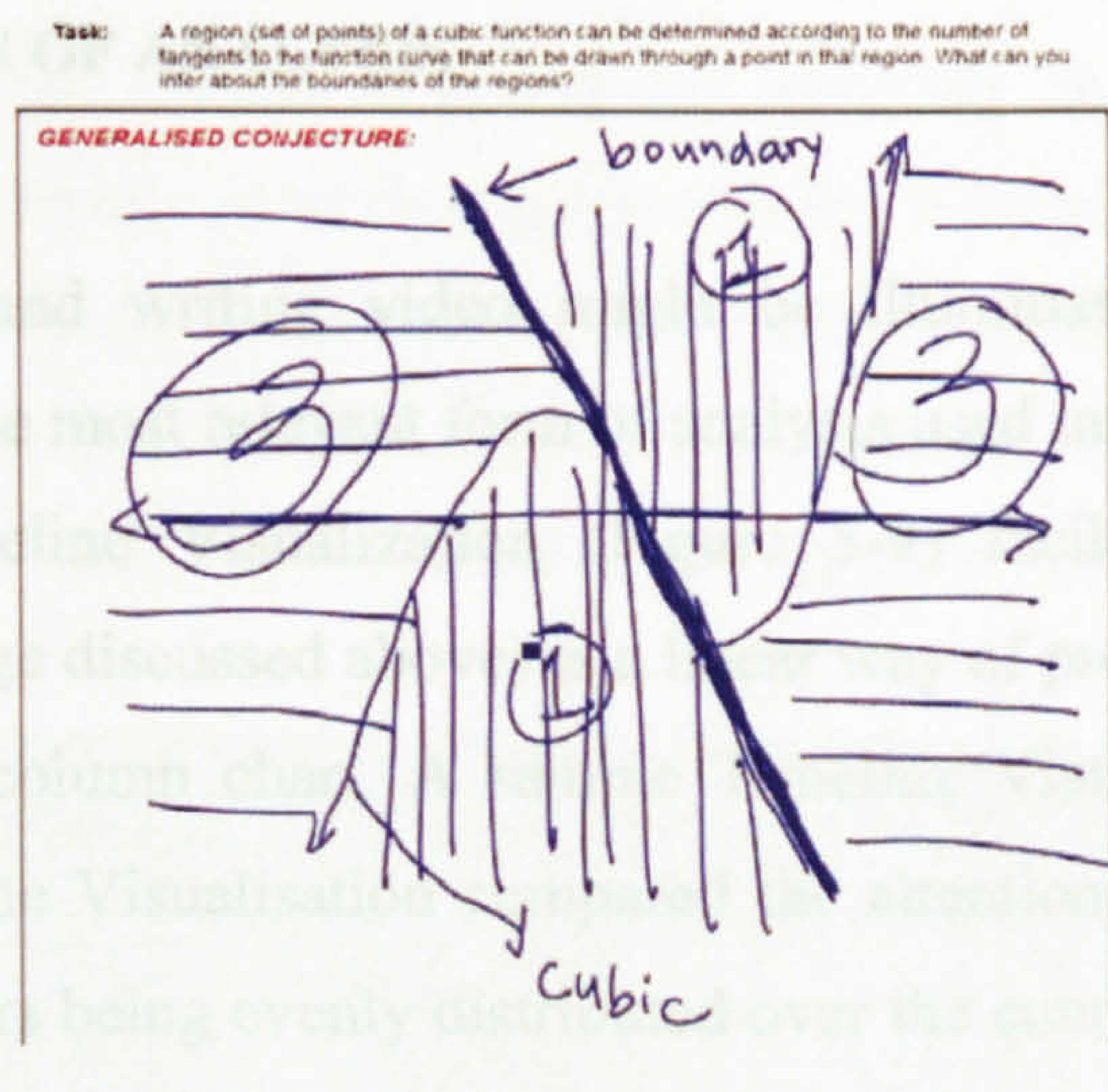


Figure 5-7 Using a sketch to explain

5.3.3 RELATING THE GAZE VIDEO AND THE WRITING VIDEO

When people can no longer hold information in their working memory, they may resort to recording information on paper. The moments when they record something might therefore provide insights into where information overload may be happening (see Figure 5-8).

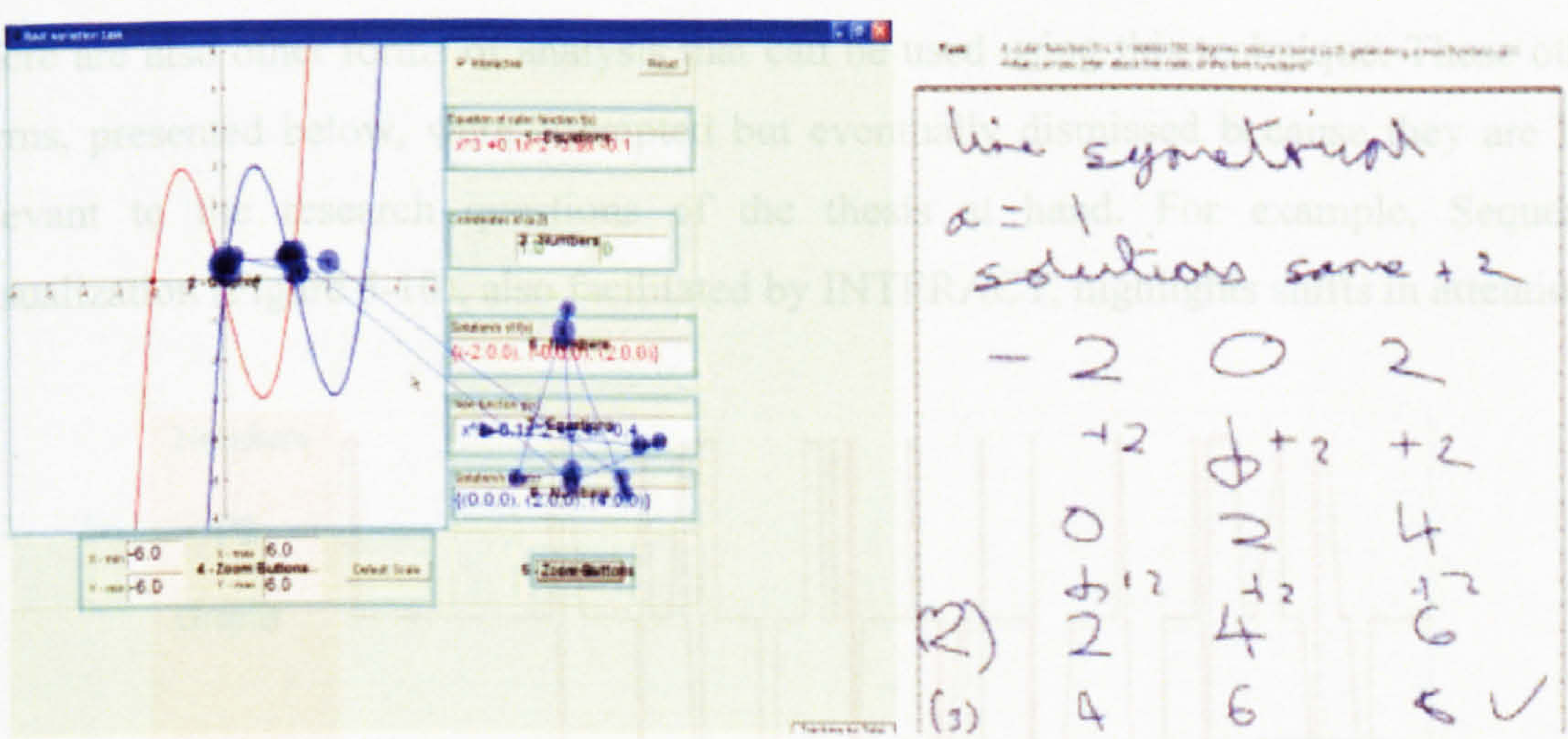


Figure 5-8 Relating gaze video and writing video

5.3.4 OTHER FORMS OF ANALYSIS

Relating gaze, action and writing video might be illuminated by further forms of quantitative analysis. The most relevant form of analysis used in the study is the Timeline visualisation. The Timeline Visualization (Figure 5-9) facility in INTERACT (the analysis software package discussed above) is a linear way of presenting a series of coded events over time in a column chart. A sample Timeline Visualisation is presented in Figure 5-9. The Timeline Visualisation compared the attention of two participants (e.g. P2's attention to numbers being evenly distributed over the course of the task whilst P1's attention to the numbers was heavy near the start).

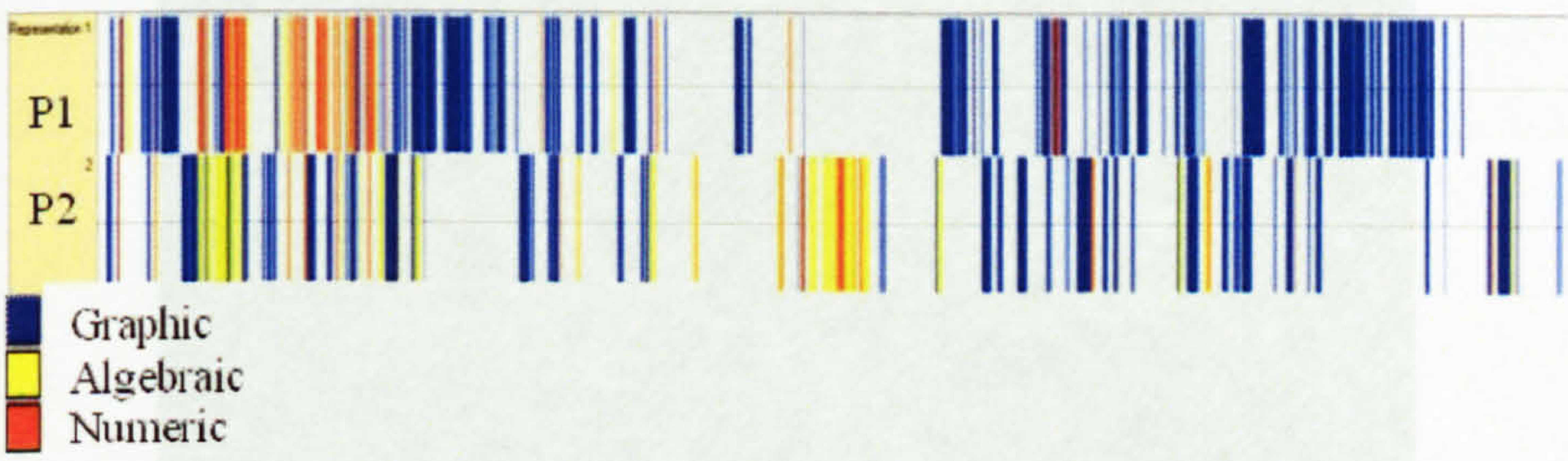


Figure 5-9 Timeline Visualization

There are also other forms of analysis that can be used using this technique. These other forms, presented below, were attempted but eventually dismissed because they are less relevant to the research questions of the thesis at hand. For example, Sequence Visualization (Figure 5-10), also facilitated by INTERACT, highlights shifts in attention.

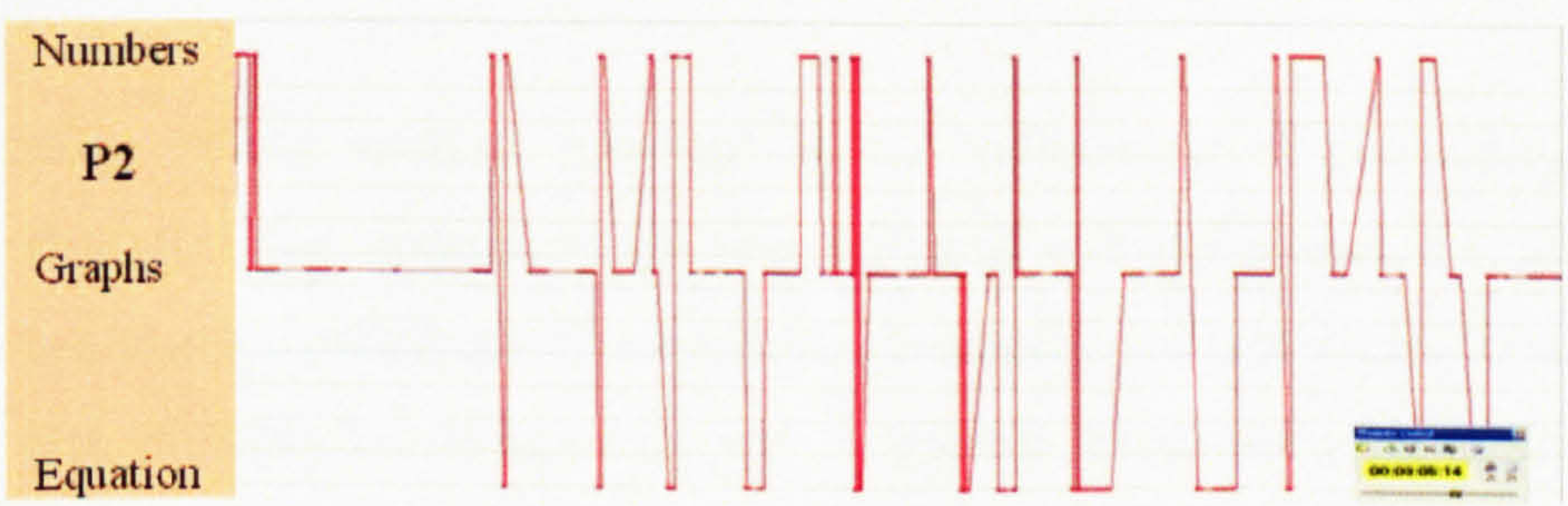


Figure 5-10 Sequence Visualization

Hotspot Visualization (e.g. Figure) is useful for relatively stable screen images, as it summarises where participants' attention appears to be focused. In this study of learning from multiple representations, for example, the Hotspot Visualization suggests that in solving the particular problem, one participant (P1) focused mostly on the graphs, with some attention to the numbers, but very little on the algebraic equations.

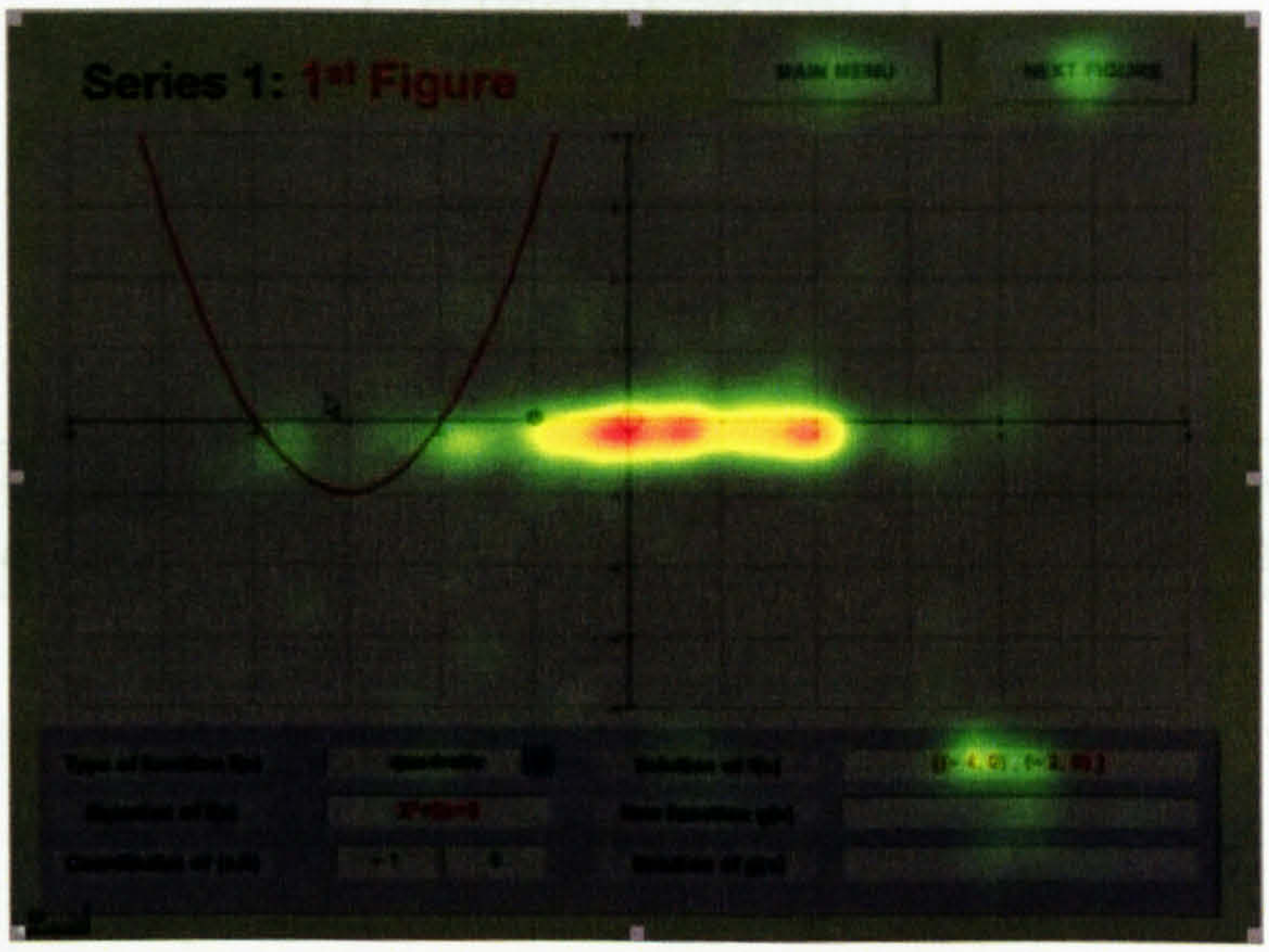


Figure 5-11 Hotspot visualization

5.4 THE PHASES OF ANALYSIS

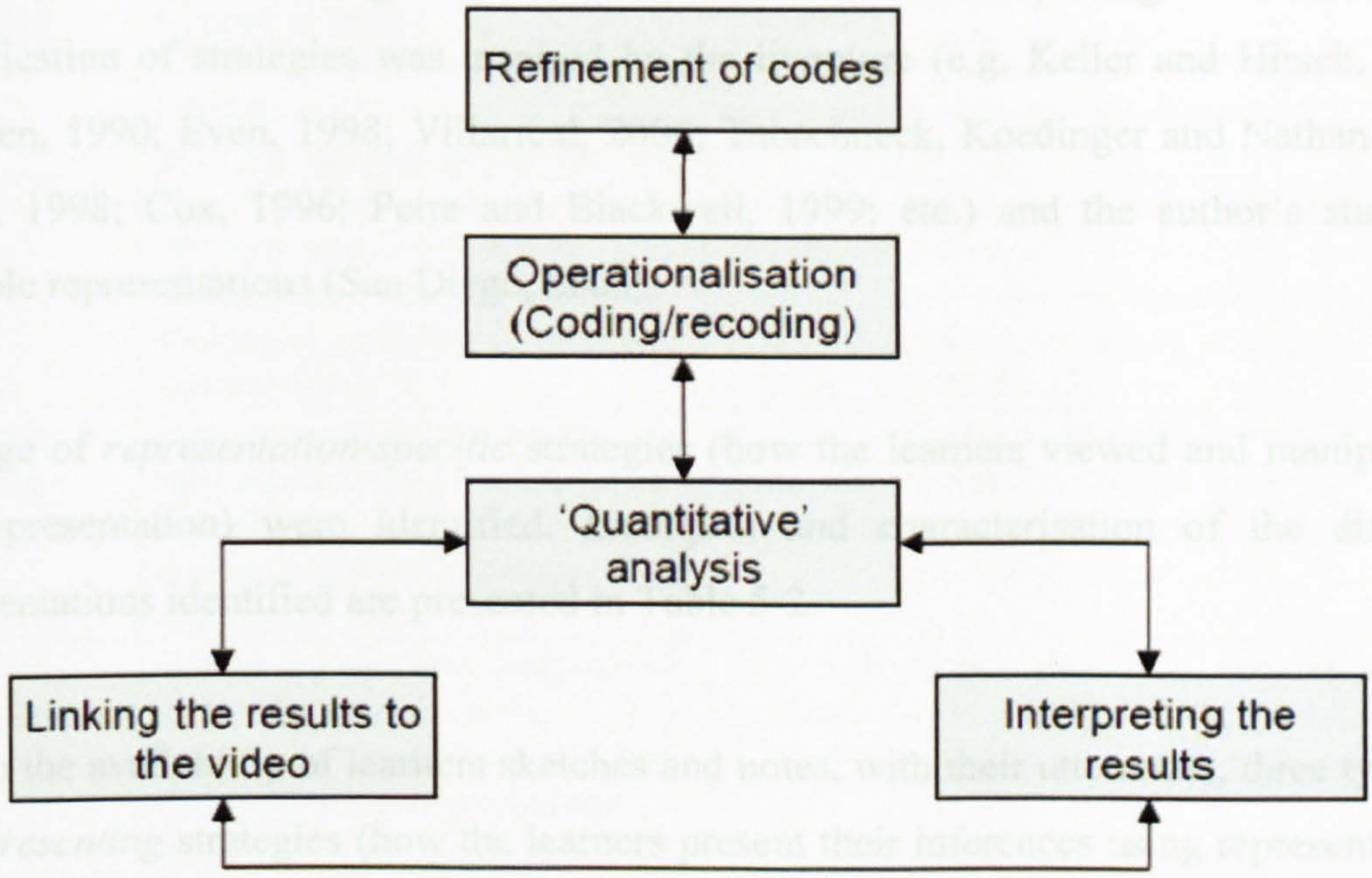


Figure 5-12 Phases of analysis

The analysis phases were based on two analytical models for video data, Jacobs et al. (1999) and Powell et al. (2003). Each recommended a model to analyse video data in non-linear phases. Based on these two models, and the nature of the data collected and the new capabilities of state-of-the-art software tools, the analysis involved iterative analytical phases (Figure 5-12).

5.4.1 REFINEMENT OF CODING SCHEME

The first phase of the analysis involved is the 'refinement of the coding scheme.' Learners' strategies for understanding and utilising representations are identified in relation to the research questions. The aim of the research is to investigate learners' use of strategies with multiple representations and to provide explanation as to why learners change their strategy. The classification of *representation-specific strategies, imagining strategies and re-representing strategies* was a convenient way of categorising the strategies identified to address these research questions. The strategies were refined by repetitive re/watching of the data. Utterances were not transcribed word for word; rather, the gaze, action, and writing videos were watched simultaneously using INTERACT. The classification of strategies was inspired by the literature (e.g. Keller and Hirsch, 1998; Ruthven, 1990; Even, 1998; Villarreal, 2000; Tabachneck, Koedinger and Nathan 1994; Aczel, 1998; Cox, 1996; Petre and Blackwell, 1999; etc.) and the author's study on multiple representations (San Diego, 2003).

A range of *representation-specific* strategies (how the learners viewed and manipulated the representation) were identified. Examples and characterisation of the different representations identified are presented in Table 5-2.

Due to the availability of learners sketches and notes, with their utterances, three types of *re-representing* strategies (how the learners present their inferences using representations not made available to them) were also identified (Table 5-3).

Table 5-2 Representation-specific strategies scheme

Strategies	Characterisation	Examples
Point-wise	Utilises discrete points of a function plotting, reading, projecting, etc.	*"As you move across this maxima... you get one going in the opposite direction" (P1 ST, 00:11:44:15)*
Graph-wise	describes and relates function's property and or behaviour Comparing shapes or contours looking at a number of graphs several times	*"As the function goes vertical it follows it" (P5 IC, 00:04:45:21 "Ah ok... I see what the rotation means now... Your point not moves to three..." (P15 SR, 00:00:32:17)*
Algebraic-chunking	interprets and relates part(s) of (an) algebraic expression (s) or equation(s) Comparing coefficients or algebraic terms	*(Whilst changing the coefficient) "It is gonna be the same no matter what the coefficient is" (P16 DR, 00:10:32:13)*
Algebraic-manipulation	Works or operates on algebraic expressions or equations Translates a problem statement to algebraic assignments and equations	*"Let us be formal about this... let g one of x is equals... err... " (P8 DR, 00:15:44:15)*
Graphic-algebraic-trial	Links graphic and symbolic forms to construct a precise symbolisation for the information available in the given graph	*(Whilst looking at the equation and the graph) "It has given me three equations even though there are only two tangents... I guess there is a line that does that..." (P7 DT, 00:01:27:10)*
Algebraic-graphic-trial	modifies a symbolic expression in the light of information gained by comparing successive expression graphs	*(Whilst entering some equations and looking at the graph) "It doesn't matter which cubic function I put in... if I rotate it by (0, 0) one of the roots will always be the same..." (P10 DR, 00:02:52:09)*
Numeric-trial	Operates on numerical representations Compares value of sets of numbers	*(Looking at the numbers written on his Tablet) "I've got these roots written down... they have been increased by two...minus three has turned into five" (P18 IR, 00:14:48:01)*
Numeric-algebraic trial	Finds pattern on numbers to construct a symbolic rule	*"The roots have shifted by two times a... two a plus one... Uhmm!" (P16 DR, 00:11:56:00)*
Graphic-numeric-trial	Links the graph with the numbers	*"... the coordinates and the midpoint and relates it to the screen (looking at the graph)... to understand what they all mean..." (P2 DT, 00:00:59:12)*

Table 5-3 Re-representing strategies scheme

Strategies	Characterisation	Examples
Visual	Presents conjectures into alternative visual forms	*(Drew some lines on the Tablet PC representing some chords extending to infinity) "The highest and lowest it can go depends on the minima and the maxima... but if it tends to infinity... that's gonna keep stretching the midpoint up and up..." (P1 SR, 00:12:40:12)*
Symbolic	Presents conjectures into alternative algebraic forms	*(Whilst writing on the Tablet PC) "a not plus... so these are the roots x, y, z..." (P6 IR, 00:22:11:00)*
Textual	Presents conjectures into textual/numeric forms	*(Writing numbers on the Tablet PC from the graph) "They are all on the x axis... one at minus three... one at minus one and one at zero... " (P5 IT, 00:05:47:06)*

Table 5-4 Imagining strategies scheme

Strategies	Characterisation	Examples
pen	Imagines a behaviour of a graph by sketching	*(Sketching on the Tablet and using the pen to showing how the point moves) ".. As the midpoint moves this point goes..." (P10 IC, 00:11:40:00)*
Mouse	Imagines a behaviour of the graph by moving the mouse	*(moving the mouse pointer) "The vertex of the graph and... I suppose it goes on to infinity... although would I get one there... I suppose I'm imagining if I will get a midpoint in here..." (P12 IC, 00:02:40:17)*
Gaze	imagines a behaviour of a certain graph by moving their eyes	*"I continue to check visually where midpoint could lie..." (P14 SC, 00:12:48:07)*
Gesture	Uses hands to imagine a behaviour of a certain graph	*(Whilst moving the hands says) "Err... the midpoint lies... ok so we got the midpoint...at this particular case at the turning point there... but you can draw anywhere P11 (saying his own name)" (P11 SC, 00:06:35:01)*
Mental	Constructs a mental representation and verbalises them	*(373 is cannot be found on the screen) "I'm trying to imagine how high the graph could be to reach 373" (P9 IT, 00:01:33:01)*

Table 5-5 Generic strategies scheme

Strategies	Characterisation	Examples
Guess-and-test	Makes a reasonable guess; tests it and revises it if necessary.	*"Actually it might not work... hmm... I'm going to investigate them..." (P12 DT, 00:08:19:09)*
Trial-and-error	Performs testing of random action	*(Inputting coordinates of x) "I'm trying to vary x... I'm trying to find the other boundary..." (P11 DT, 00:21:48:17)*
Working-backward	Works with a certain result then operates on them	*"Coordinate of zero adds on 2... coordinate of two adds on 4 so three it adds on six it seem to be doing double..." (P3 DR, 00:06:01:10)*
Working-with-preferred-cases	Uses a familiar representation such as a simple equation or smaller values, or a simple graph	*"I'll start with something simple x^3" (P4 DR, 00:00:12:05)*
Sketching	Constructs a diagram	*(Drawing a graph) "So the point is always inside... ok" (P5 IC, 00:08:26:22)*
Recording	Performs an action to support memory	*(wrote (a,0) equals zero) "shift equals two..." (P6 IR, 00:06:16:07)*

Table 5-6 Interface-supported strategies scheme

Strategies	Characterisation	Examples
Random-interaction	views/animates/interact several graphs randomly	*"Now I'm gonna try completely random roots and see what it does..." (P18 IR, 00:04:12:23)*
Systematic-interaction	Views/animates/interacts graphs at a specific sequence	*"If I choose a point one zero... we have two point of tangency... at zero one... shows the same thing..." (P4 DT, 00:02:10:13)*
Shared-characteristic-interaction	Compares an identified figure with other figures	*"I'm gonna try this out again by doping another function...to see if it holds for the next one..." (P3 DC, 00:08:37:13)*

Further qualitative analysis of gazes with actions and writing allowed identification of five imagining strategies (how the learners used mental visualisations as far as can be detected from the data). Examples are given in Table 5-4.

Other strategies that were also identified are generic strategies (how the learners carry out the task completion) and interface-supported strategies (how the learners interact with the interface of the software used in the study). These strategies are shown in Table 5-5 and, Table 5-6 respectively.

5.4.2 OPERATIONALISATION (CODING/RECODING)

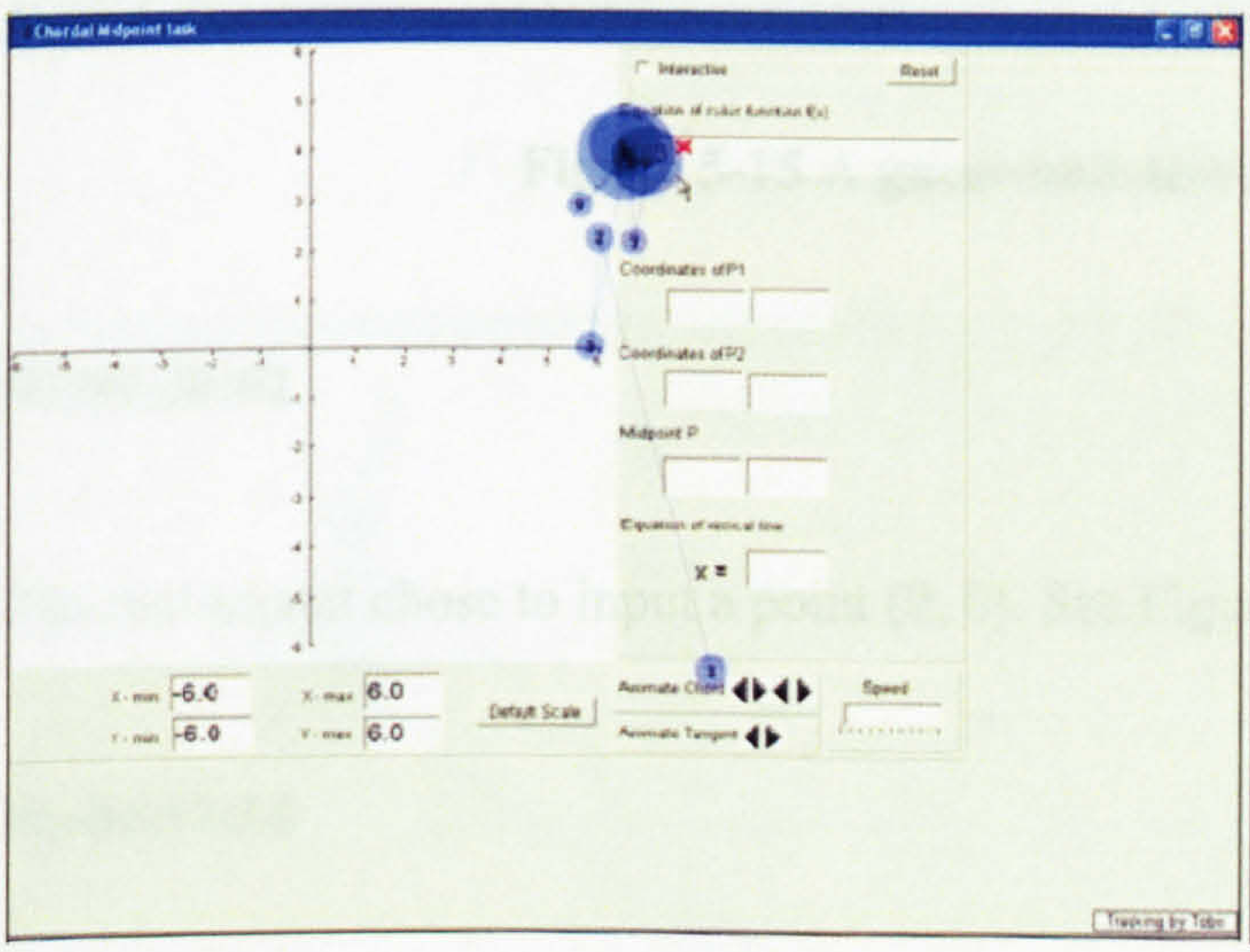
Representation-specific, imagining, re-representing, generic, and interface-supported strategies, as characterised in the previous section, can be operationally defined by relating what learners say, do, see and write as they complete a task with multiple representations.

Figure 5-13 Participant 4's 'action video' snapshot

A verbatim transcription of one participant's task with video snapshots is presented below. This is an illustration of how the synchronous replay of talk, screen, writing and gaze data was coded. This detailed account shows the richness of the data recorded, the kinds of coding decisions taken, and the reasons for them (The data from this account are later used in the analysis chapter 7 section 7). The start and end time of each strategy was determined by means of going through each of the coded events iteratively, changing each start and end time to match the start and end of an utterance for a particular strategy. Figure 5-13 is a snapshot of Participant 4 (P4).

The task involved in this illustrative coding is the *Chord task* instantiated in *Dynamic form*. In this type of software, the participants can input equations and numbers. As a reminder of the *Chord task*, the participant is asked to make inferences about the boundary of the set of midpoints that can be drawn on any cubic function. The experiment took place in 17 October, 2005. It took the participant twenty-one minutes (21 min.) to complete the task. The *chord task* was the third of the tasks that the participant did at that time. The participant also did the other two tasks in *Dynamic* form.

For *Dynamic* instantiation, any participant can enter any equation that comes into her mind. They can choose to consider the contour of the graph; the shape of the graph that the equation represents (e.g. a cubic can have two turning points or one). They can also consider the nature of the cubic equation (e.g. 'simple equation' or 'complex equation').

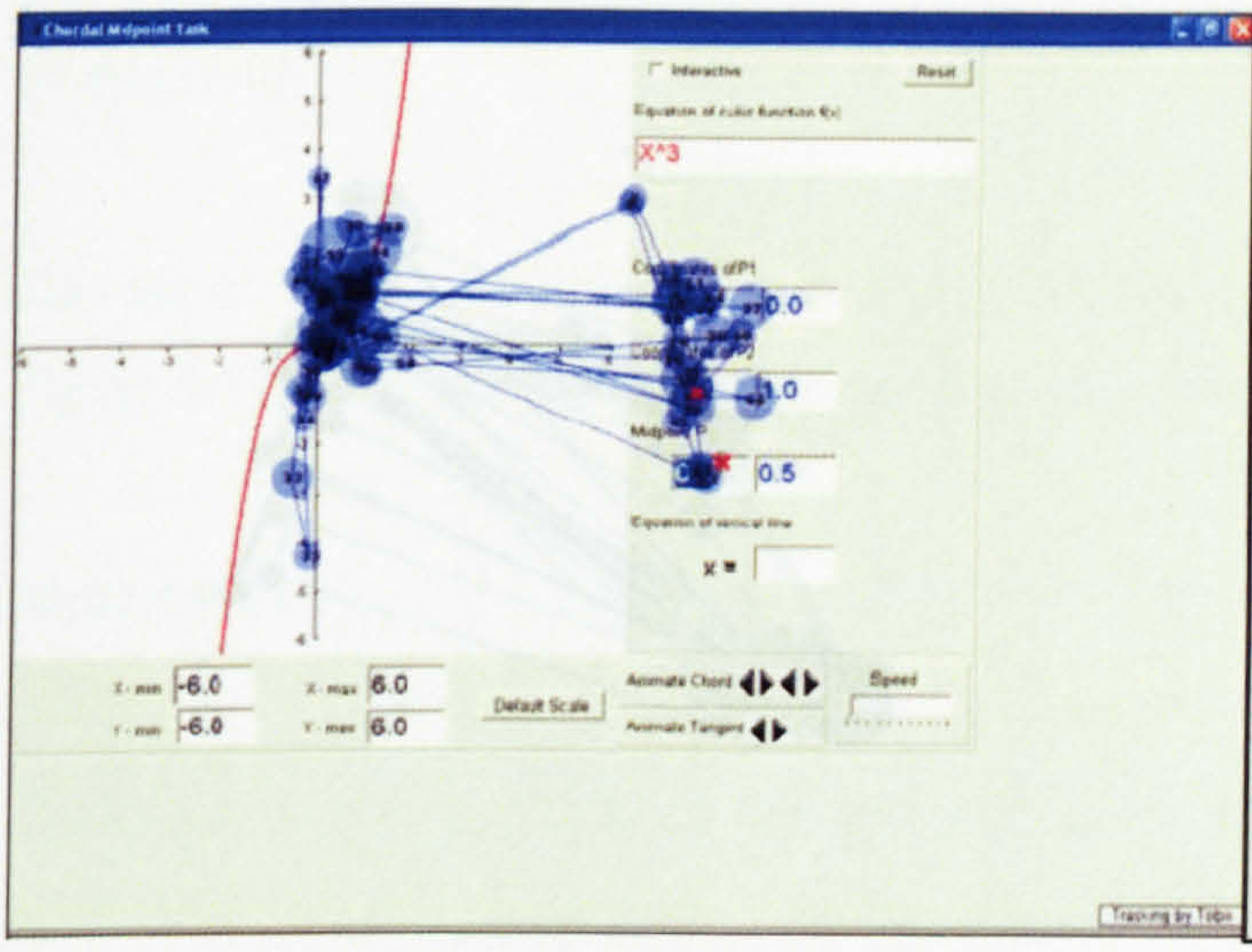


Code: working-with-preferred-case (generic strategy)
(P4 started to talk, gazed at the equation text box and typed in x^3)
(00:00:08:12)
"Again I always like to do simple stuff so I gonna go x^3 and see what happens there..."

Figure 5-14 P4's working-with-preferred-case strategy

00:00:08:12

P4 entered an equation. She chose to enter a 'simple' equation ("x^3"). That it was a deliberate decision to enter a simple cubic is supported by her utterance (Figure 5-14). This event was therefore coded as a *working-with-preferred-case strategy*.



Code: guess-and-test (generic)
(P4 typed in numbers. She gazed at the graph and the numbers back and forth)
(00:00:20:02)
"I'm going to test and see what happens if I make a line zero... just the x-axis I need to put in right? And let's say zero zero... "
(00:00:37:24)
"I don't think I'm gonna have anything happening... alright let's see one two...one one whatever..."

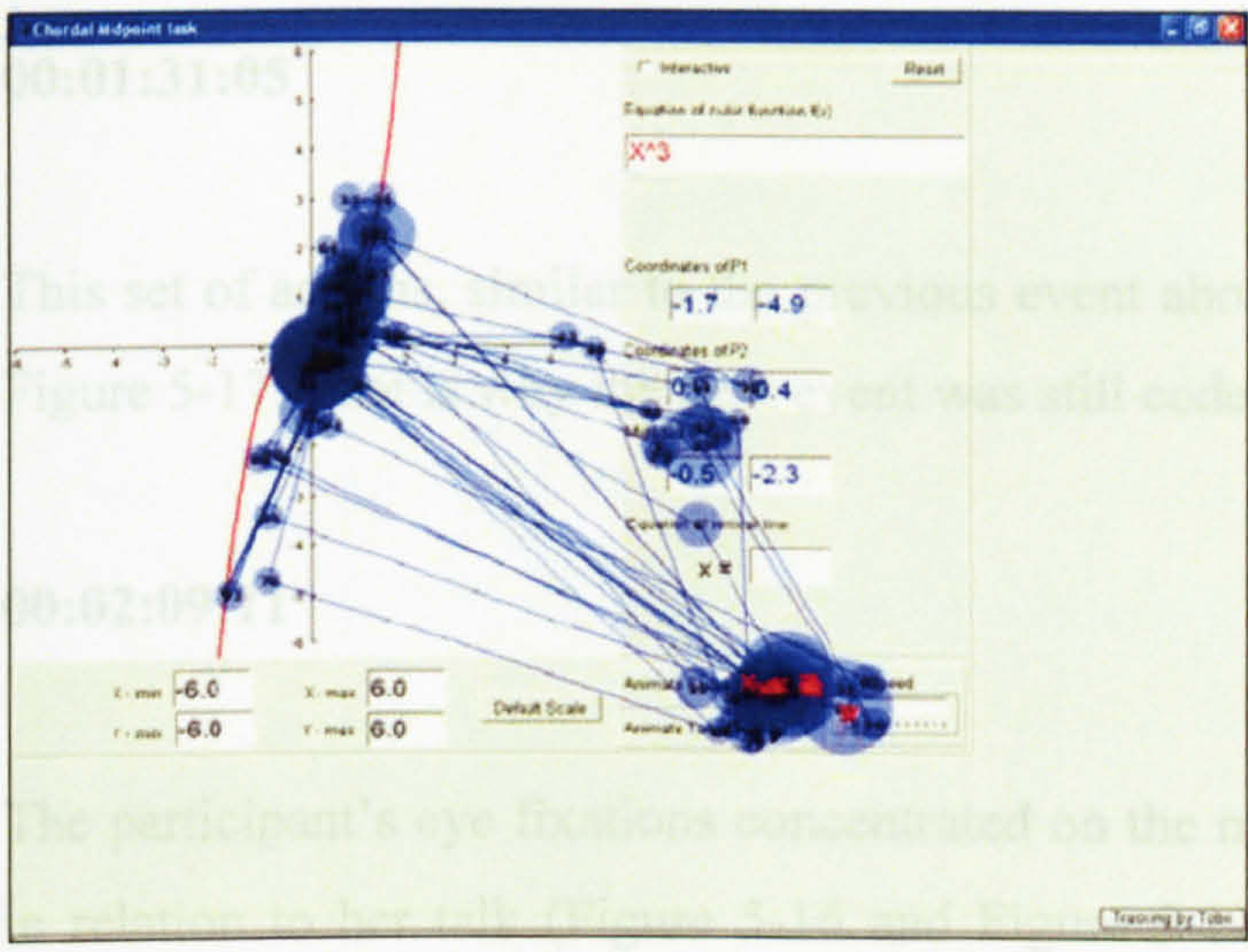
Figure 5-15 A guess-and-test strategy of P4

00:00:20:02

The participant chose to input a point (0, 0). See Figure 5-15.

00:00:37:24

P4's eye gazes showed that she spent time looking at the line she made and the corresponding numbers she entered. The 'eye-marks' (i.e. the fixations and saccades superimposed) on the screen, with her talk, signified that the line through (0, 0) was being tested. This event in Figure 5-15 was also coded as a *guess-and-test strategy* because her utterances present an obvious testing of a guess.



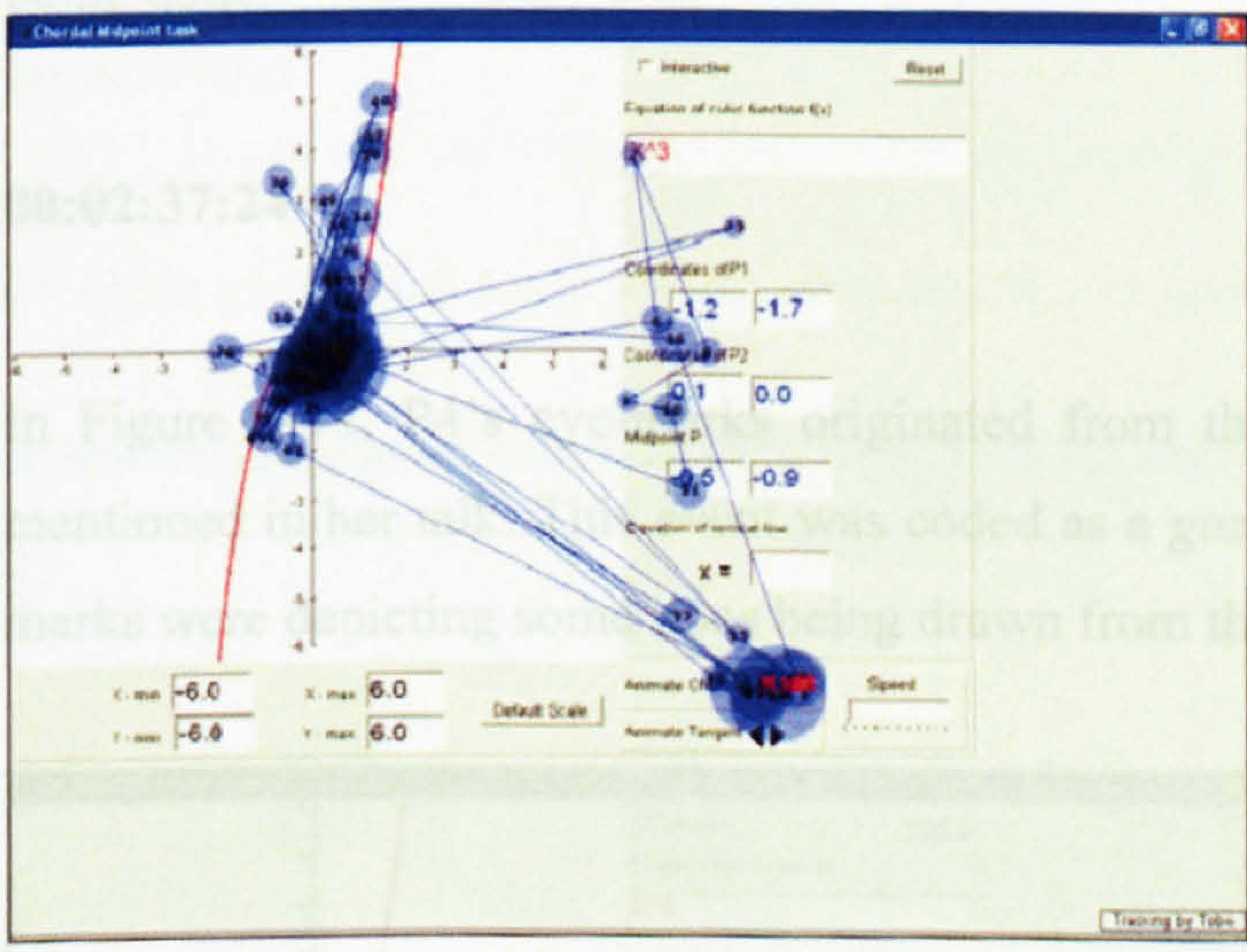
Code: random-interaction (interface-supported)

(P4 clicked the animation buttons. She animated the graph up-down, left-right, right-left, down-up, and so on. She concentrated on looking at the graph)

(00:01:24:06)

"OK... So I want to move... This is correct what I'm doing... OK that's something... If I move... This goes away to fast... How do you stop it?"

Figure 5-16 A random-interaction strategy of P4



Code: trial-and-error (generic); random-interaction (interface-supported)

(P4 read task. Then, she animated the graph up-down, left-right, down-up and so on)

(00:01:31:05)

"Ok. I lost focus of what I was doing"

(00:02:09:11)

"If... this goes... It gonna find a tangents surely"

Figure 5-17 A random-interaction and trial-and-error strategy of P4

00:01:24:06

The time spent in between mouse click operations was short. The direction of the graph's animation was 'unorderly'. Figure 5-16 shows the location of the mouse click operations, the 'red x marks' superimposed on top of the animation buttons (lower-right part of the interface). This was coded as a *random-interaction strategy*.

00:01:31:05

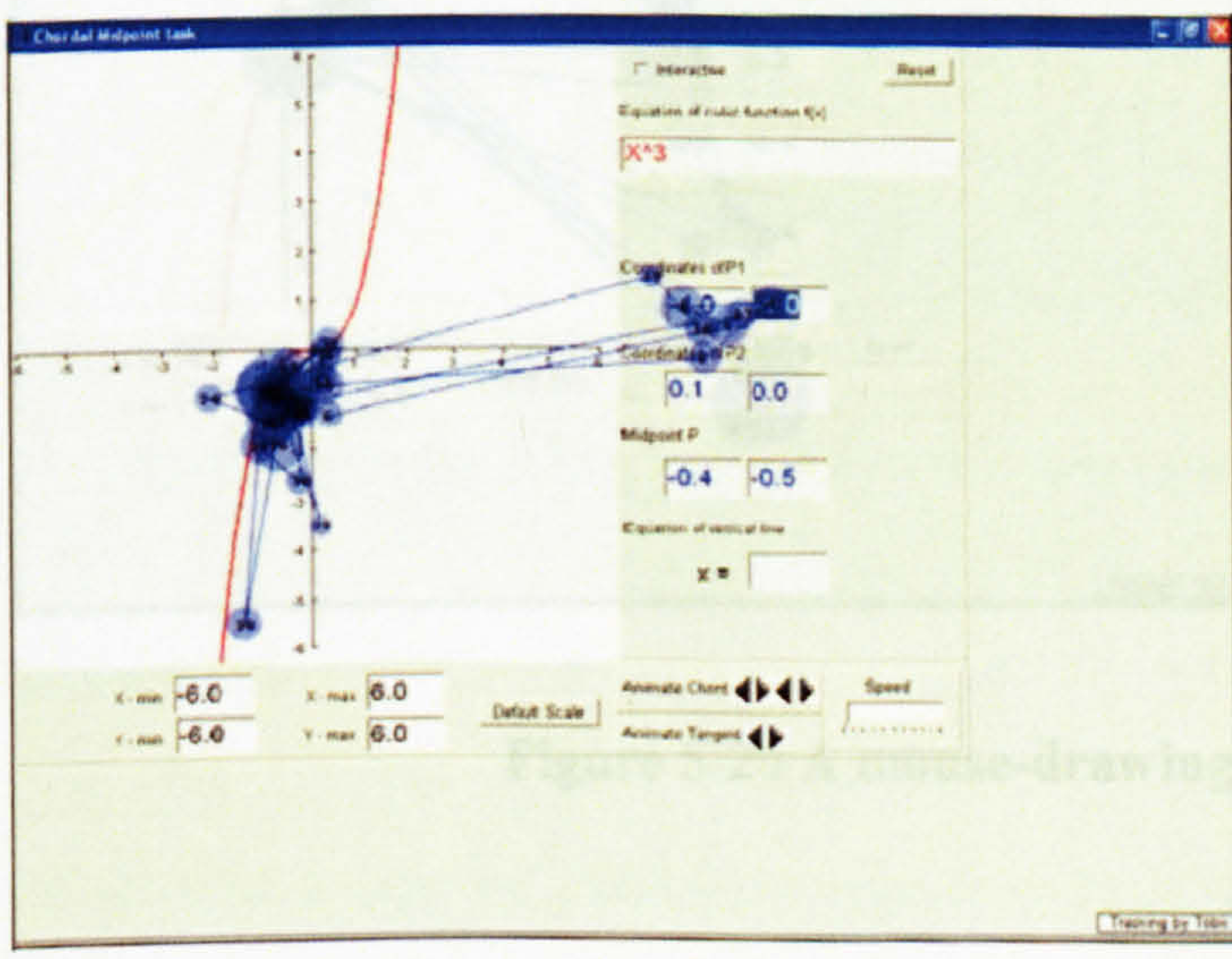
This set of actions, similar to the previous event above, continued in the same manner in Figure 5-17. That is why the next event was still coded as a *random-interaction strategy*.

00:02:09:11

The participant's eye fixations concentrated on the movement of the midpoint. P4's gaze in relation to her talk (Figure 5-16 and Figure 5-17), shows that the movement of the midpoint of the graph was under test. This event was also coded as a *trial-and-error strategy* because of the randomness of the computer operations in relation to the testing being done.

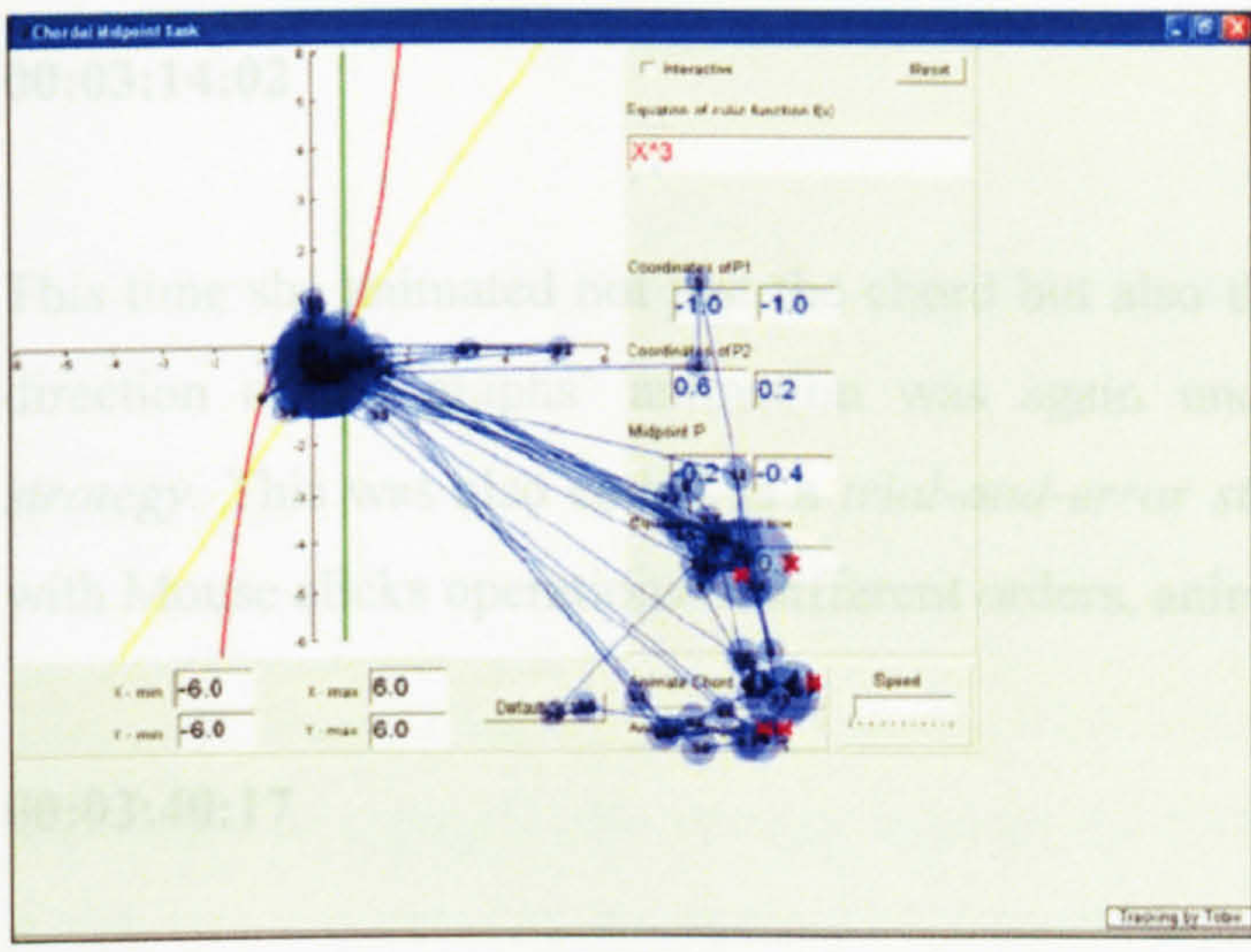
00:02:37:24

In Figure 5-18, P4's eye-marks originated from the graphical point (-1, -1) that was mentioned in her talk. This event was coded as a *gaze-drawing strategy* because the eye-marks were depicting some lines being drawn from the point (-1, -1).



Code: gaze-drawing (imagining)
(P4 looked at the graph)
(00:02:37:24)
"I'm not certain what this means..."
(00:02:57:00)
"minus 1... I'm just checking to see... which is the next point I should look at... which is minus one..."

Figure 5-18 A gaze-drawing and trial-and-error strategy of P4



Code: random-interaction; trial-and-error

(P4 animated two graphs in up-down direction, left-right, right-left, down-up, and so on. She concentrated looking at the graph)

(00:03:14:02)

"Err... Let me see what happens when I use... a tangent line... a vertical line... x equals zero is the easiest... (Inaudible)"

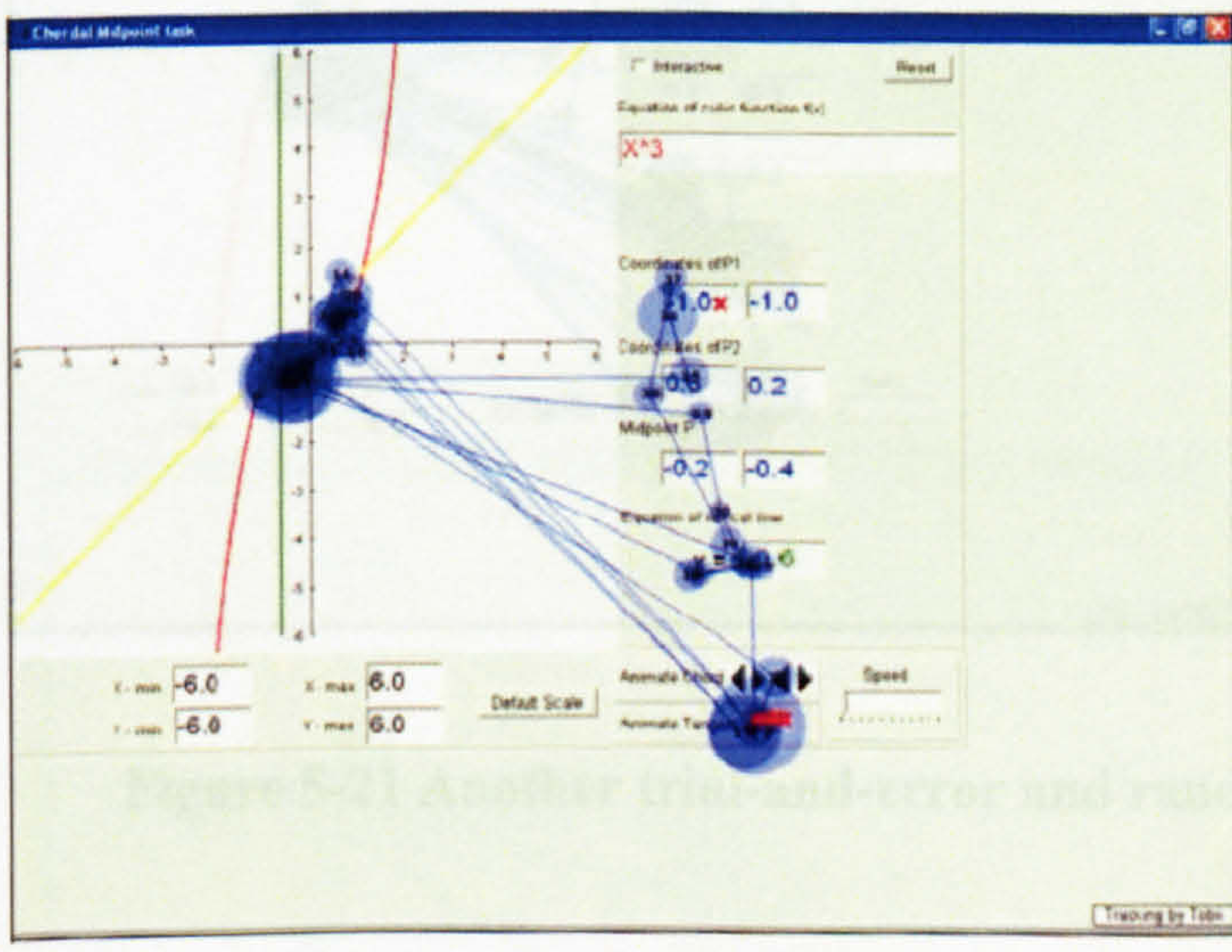
(00:03:27:19)

"Oh that's a tangent line... ok... equation of the vertical line..."

Figure 5-19 Another random-interaction and trial-and-error strategy of P4

00:02:57:00

P4 again performed some mouse click operations (see Figure 5-19).



Code: mouse-drawing (imagining)

(P4 pointed at minus zero then at point six using the mouse. Then she kept moving the mouse pointer)

(00:03:40:17)

"That's a tangent line... ok... equation of the vertical line..."

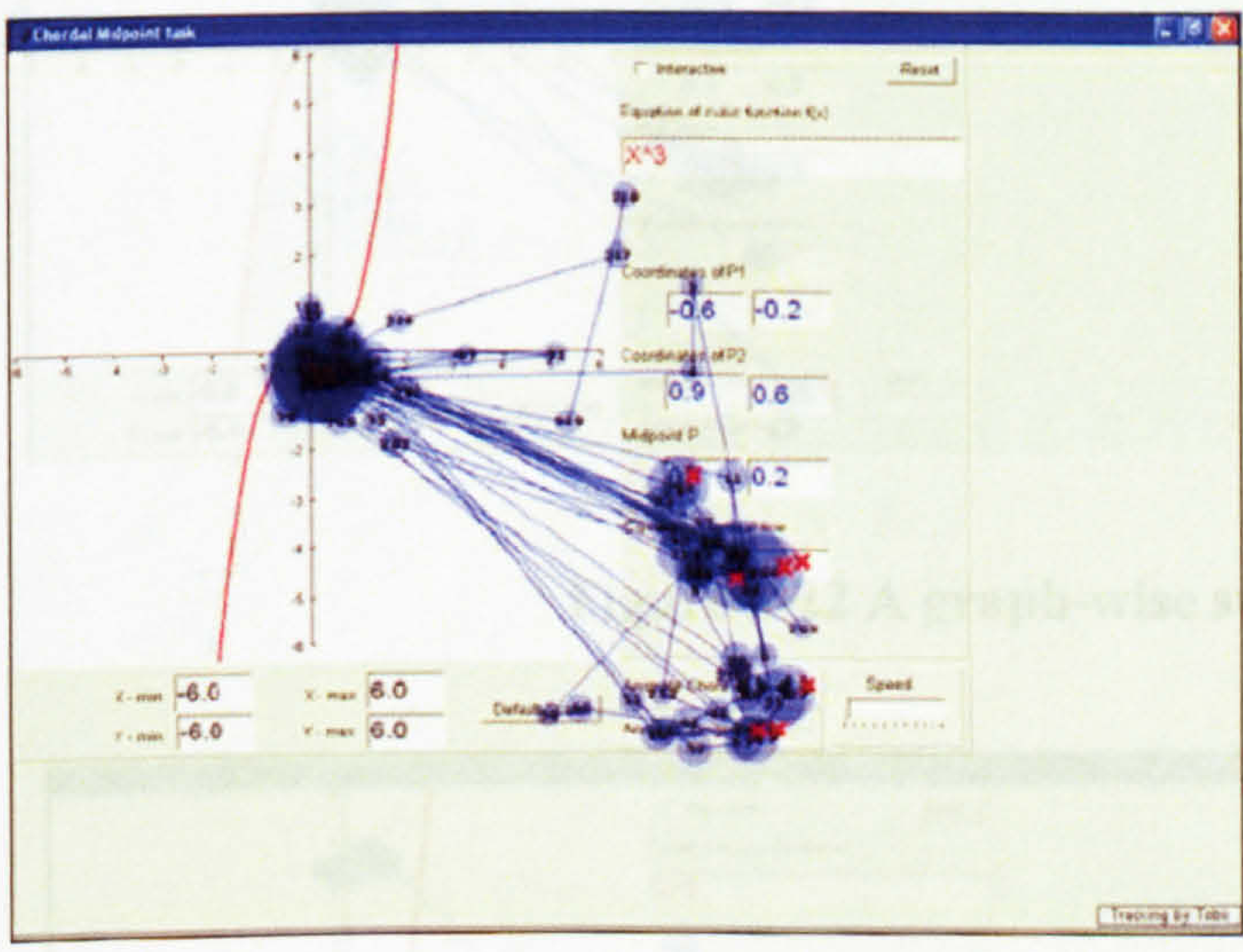
Figure 5-20 A mouse-drawing strategy of P4

00:03:14:02

This time she animated not just the chord but also the vertical line on the interface. The direction of the graphs' animation was again unorderly; thus, a *random-interaction strategy.* This was also coded as a *trial-and-error strategy* because the testing was done with Mouse clicks operations in different orders, animating the graph in every direction.

00:03:40:17

The movement of the mouse pointer depicted some lines from the point mentioned in the description of P4's action in Figure 5-20, minus zero point six (-0.6). That is why this event was coded as a *mouse-drawing strategy.*



Code: trial-and-error (generic); random-interaction (interface-supported)
(P4 changed one of the numerical inputs then again animated the graphs in every direction. Then she frustratedly gasped, and read the problem)
(00:04:34:00)
"I don't have a clue what's going on…I'm getting lost with this question…"

Figure 5-21 Another trial-and-error and random interaction strategy of P4

00:04:34:00

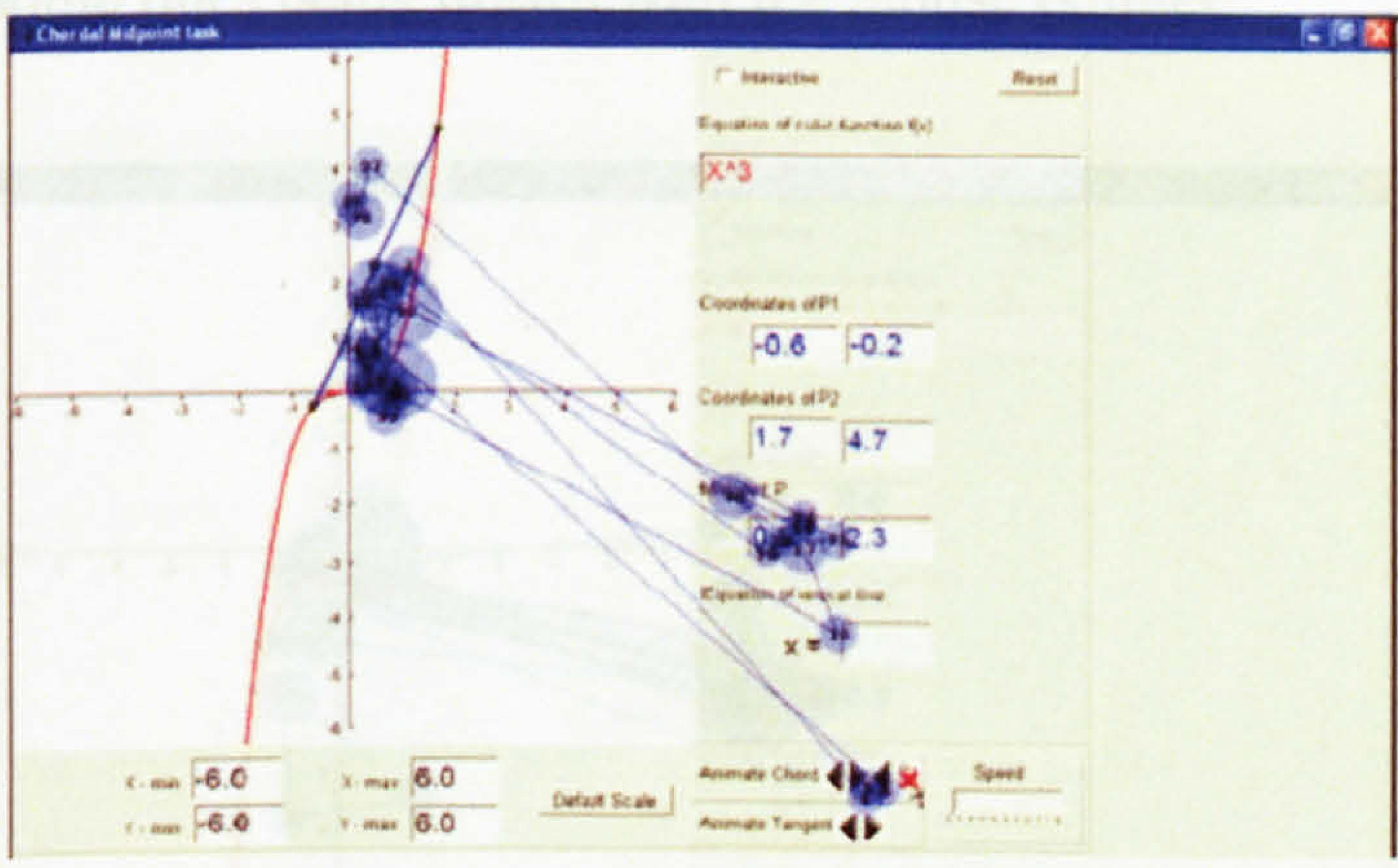
P4 showed a sign of frustration (Figure 5-21). This event was coded as a *random-interaction strategy* because she animated the graphs in an 'unsystematic' manner. Also, this was coded as a *trial-and-error strategy* because she kept on trying and testing different numerical inputs.

00:05:04:05

P4 clicked the animation button and watched the graph (Figure 5-22).

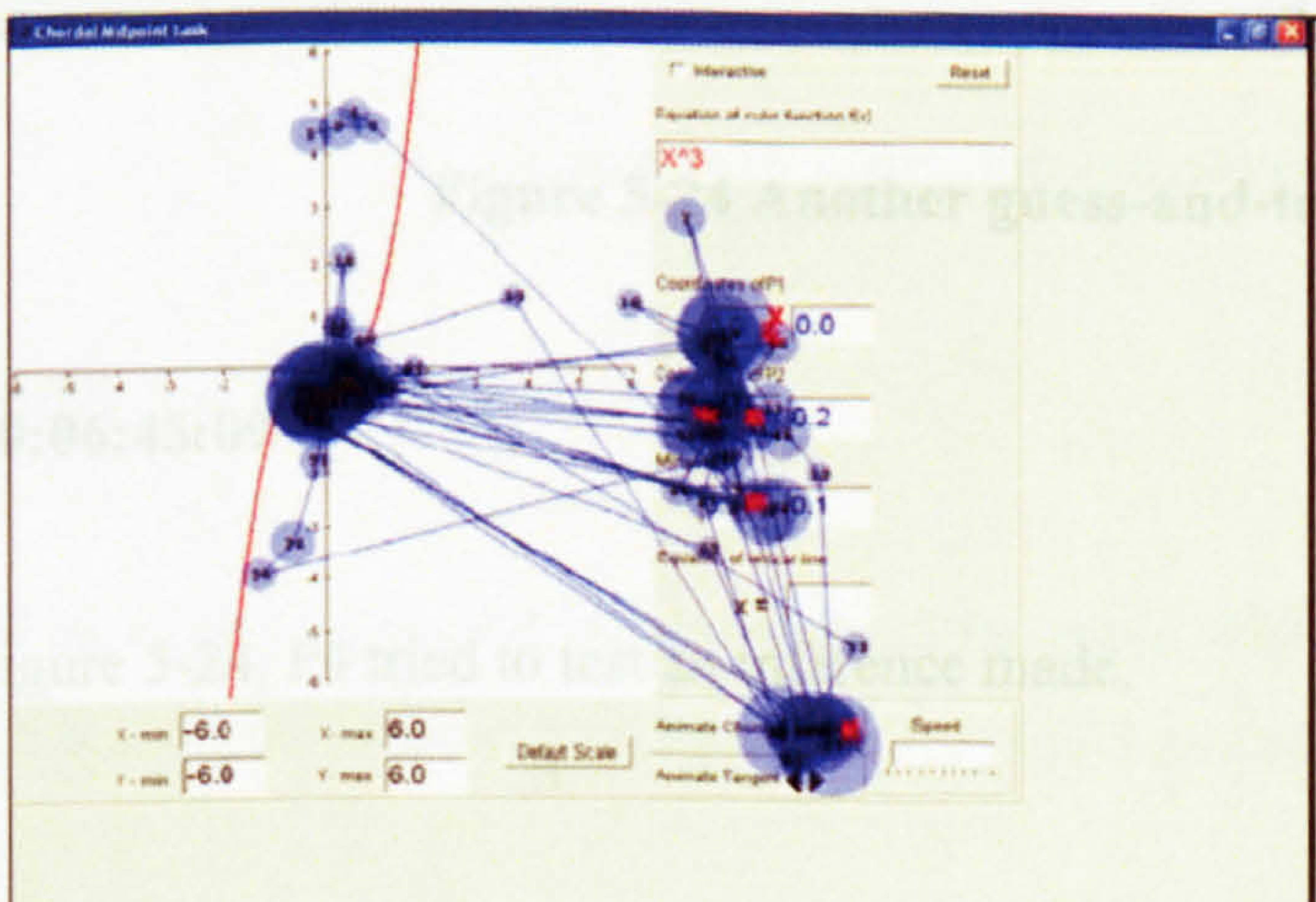
00:05:17:23

Then, she stopped the animation and described what was going on with the graph. *This was coded as a graph-wise strategy* because the fixations generally dwelled on the graph which was supported by the participant's description of the graph's behaviour.



Code: graph-wise (representation-specific)
(P4 continued animating the graph. She concentrated looking at the graph)
(00:05:04:05)
"Yeah... I know that thing... but..."
(00:05:17:23)
"The midpoint goes up... it goes up..."

Figure 5-22 A graph-wise strategy of P4

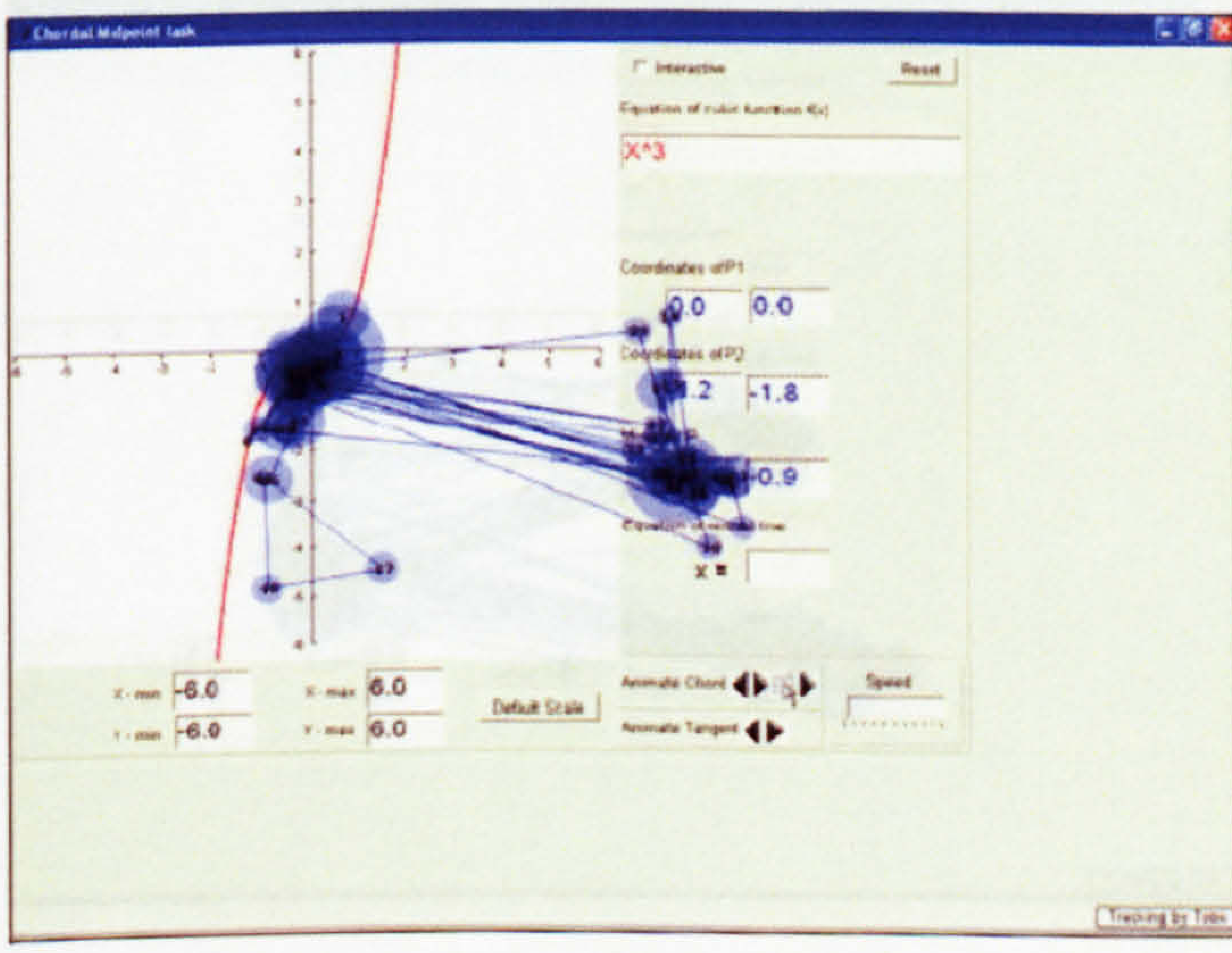


Code: trial-and-error (generic); random-interaction (interface-supported); mouse-drawing (imagining)
(P4 inputted different numbers. Then, she animated the graph. She used the mouse and pointed at (0, 0). She moved the mouse pointer downwards and upwards)
(00:05:19:17)
"Let me just try something easy... (inaudible)... I can stop that... Zero zero that's the easies... It's the region that moves with this"

Figure 5-23 A trial-and-error, random-interaction and mouse-drawing strategy of P4

00:05:19:17

In Figure 5-23, she tried another set of numerical inputs and animated the chord in a 'haphazard' manner (similar to the ones discussed above). This provided the basis for coding this event as a *random-interaction strategy.* This event received multiple codes (Figure 5-23). P4's talk mentioned the trying of *"something easy"* and checked this by animating randomly. That is why this event was also coded as a *trial-and-error strategy.* In addition, this event also received a *mouse-drawing strategy* code, because of the following actions. P4 used the mouse to point at (0, 0), mentioned in her talk. She moved the mouse pointer from the point (0, 0) to some part of the cubic graph. This showed some lines being drawn with the mouse pointer.



Code: guess-and-test (generic)
(P4 watched the graph as it animates.
(00:06:45:09)
"It goes from zero... it goes to the tangent... zero zero... and decreases"
(00:06:58:18)
"I'm just testing to see if the midpoint ever crosses... if ever crosses line here"

Figure 5-24 Another guess-and-test strategy of P4

00:06:45:09

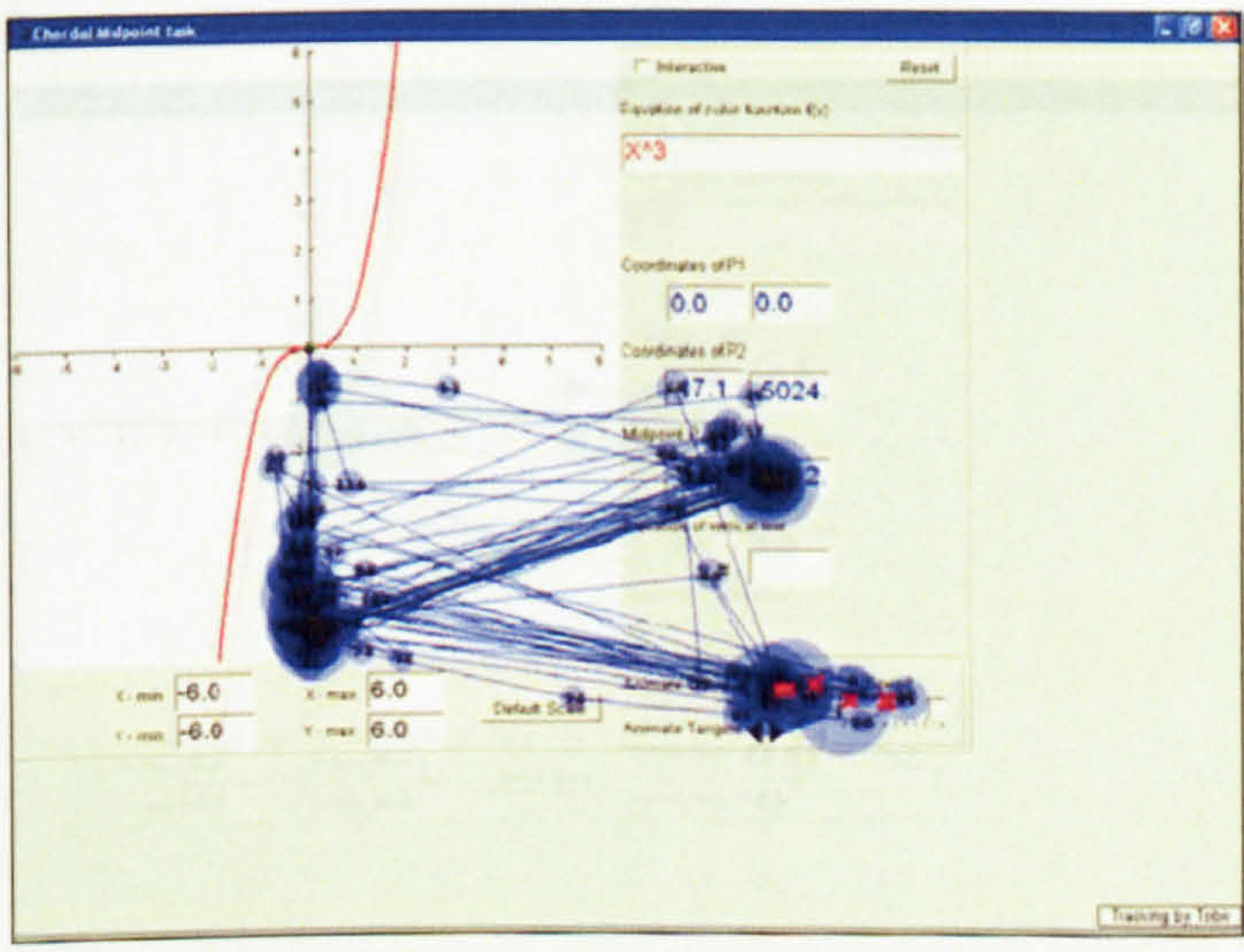
Figure 5-24, P4 tried to test an inference made.

00:06:58:18

She was checking whether the midpoint will cross the Y-axis. This was a *guess-and-test strategy.*

00:07:50:19

When P4 animated the chord, she gazed continuously at the graph and numbers (Figure 5-25). Her talk showed that she tried to make a connection between the graph and the numbers. This was coded as a *graphic-numeric strategy*.



Code: graphic-numeric (representation-specific); (00:07:50:19)

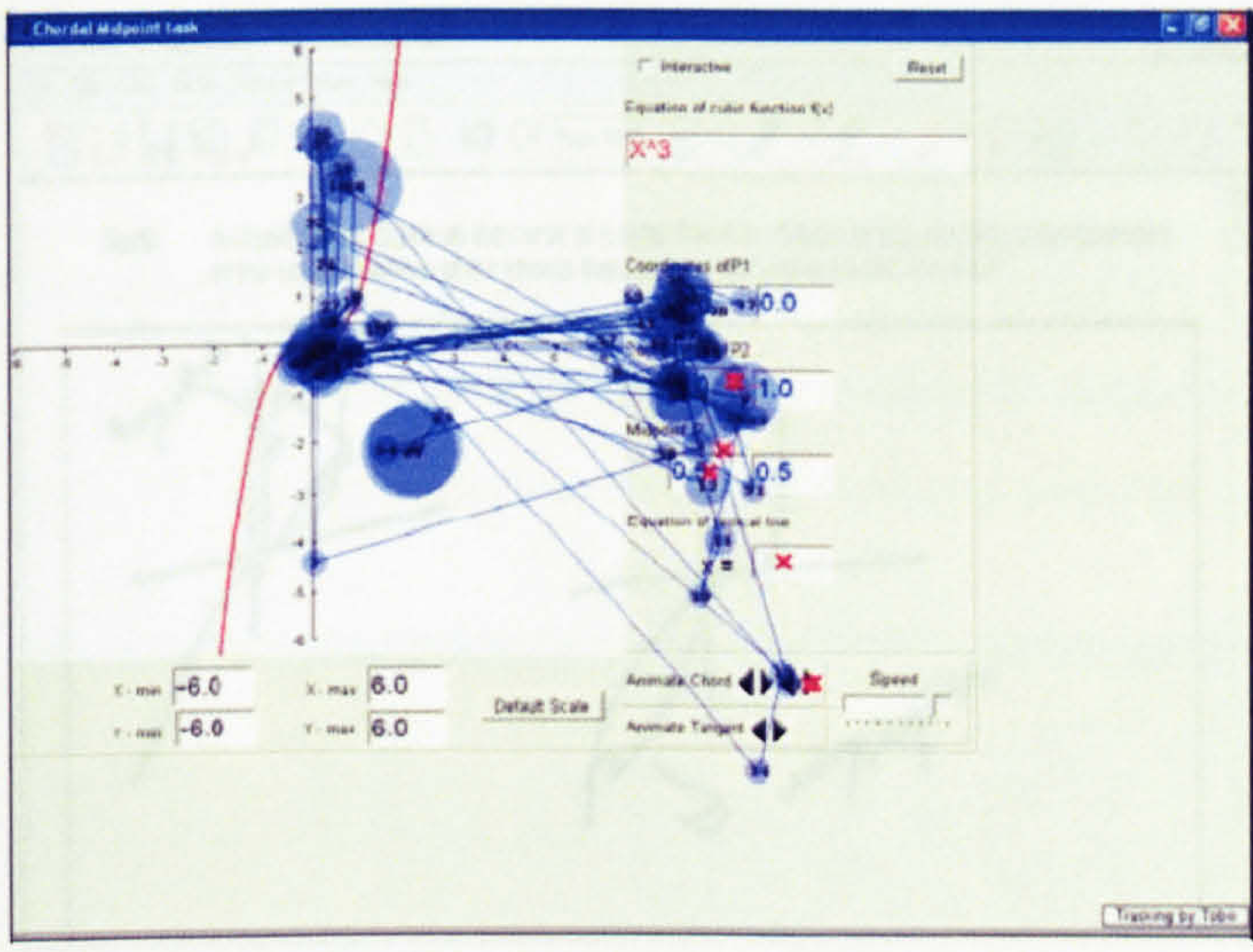
(P4 animated the graph again. The graph moved downwards. She looked at the graph and the numbers back and forth as she watched the animation.

"If I increase the speed maybe I'll know... I don't know it looks... A bit likes... asymptotic to it... (Inaudible) ...whether it's gonna be asymptotic to it... it is not moving out... right..."

Figure 5-25 A graphic-numeric strategy of P4

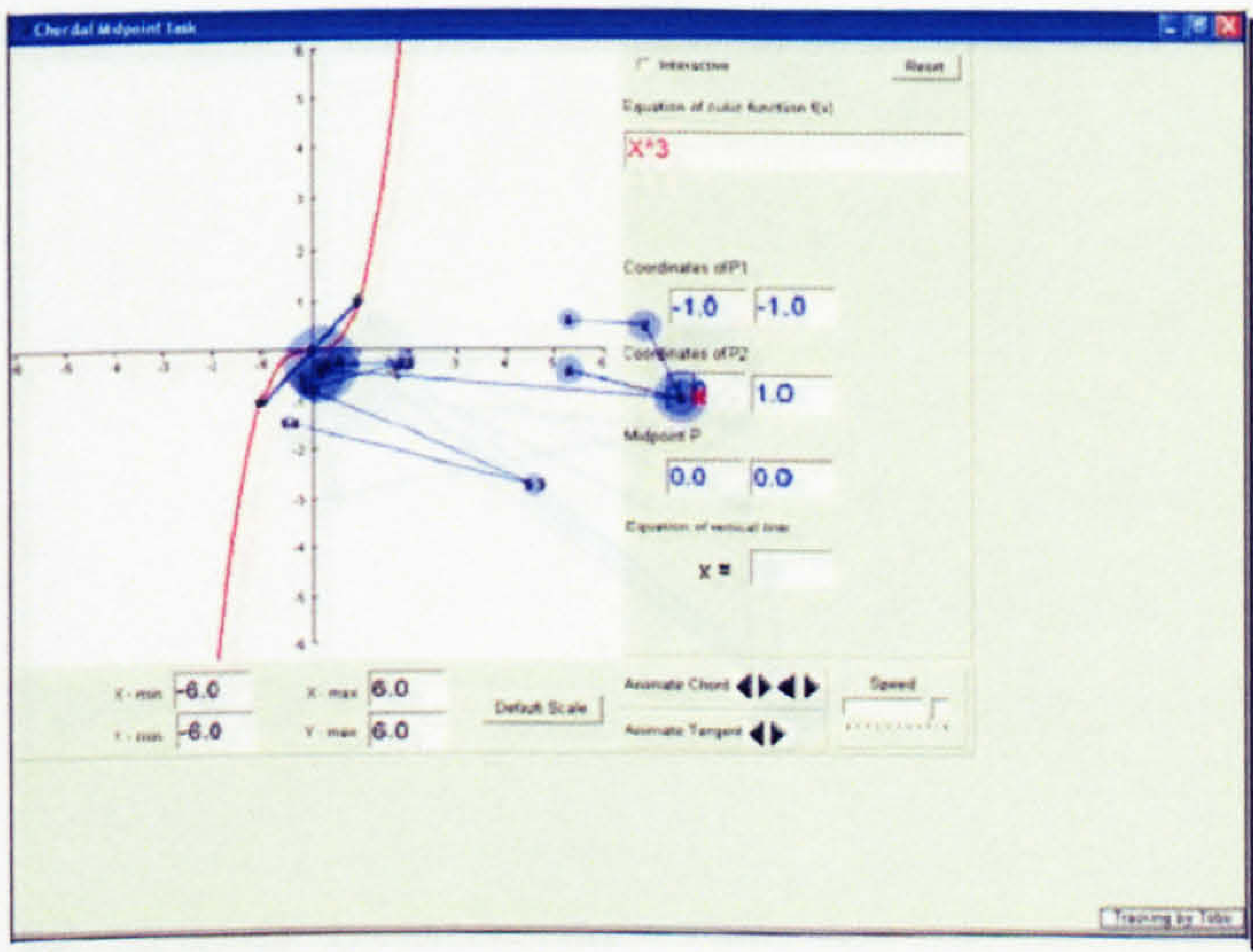
00:08:26:16

P4 guessed that on the bottom part of the cubic (Figure 5-25 shows this eye-marks) and the upper part of the cubic (Figure 5-26) would be the same (*guess-and-test).*



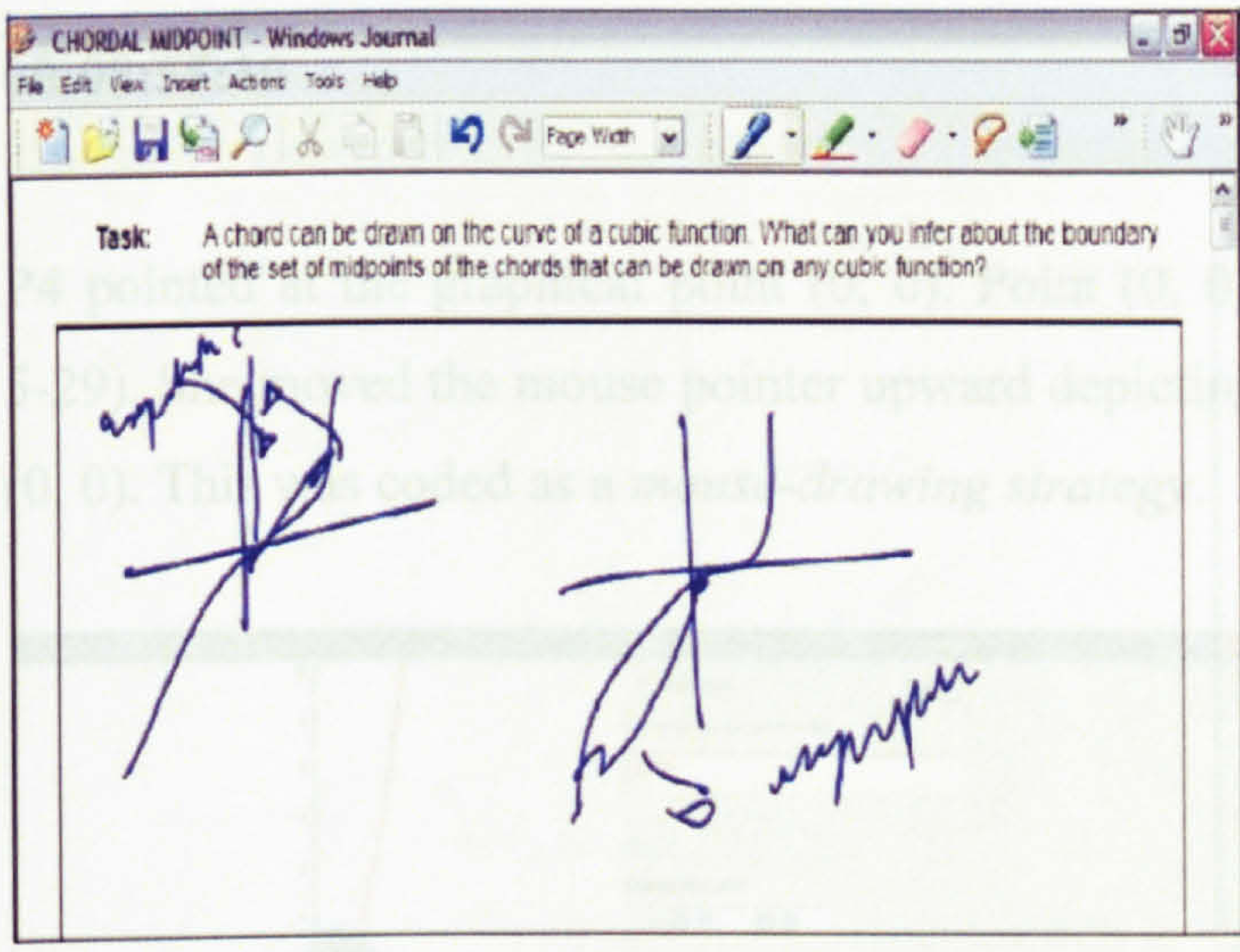
Code: guess-and-test (generic)
(P4 inputted some numbers then animated the graph)
(00:08:26:16)
"That's possibly something there... let us try one... I expect the same thing should happen. It's really asymptotic here... that's alright... err... So that's the point zero zero... if I have it on... zero zero and... and then one... (Inaudible)... If I do it from minus one..."

Figure 5-26 Another guess-and-test strategy of P4



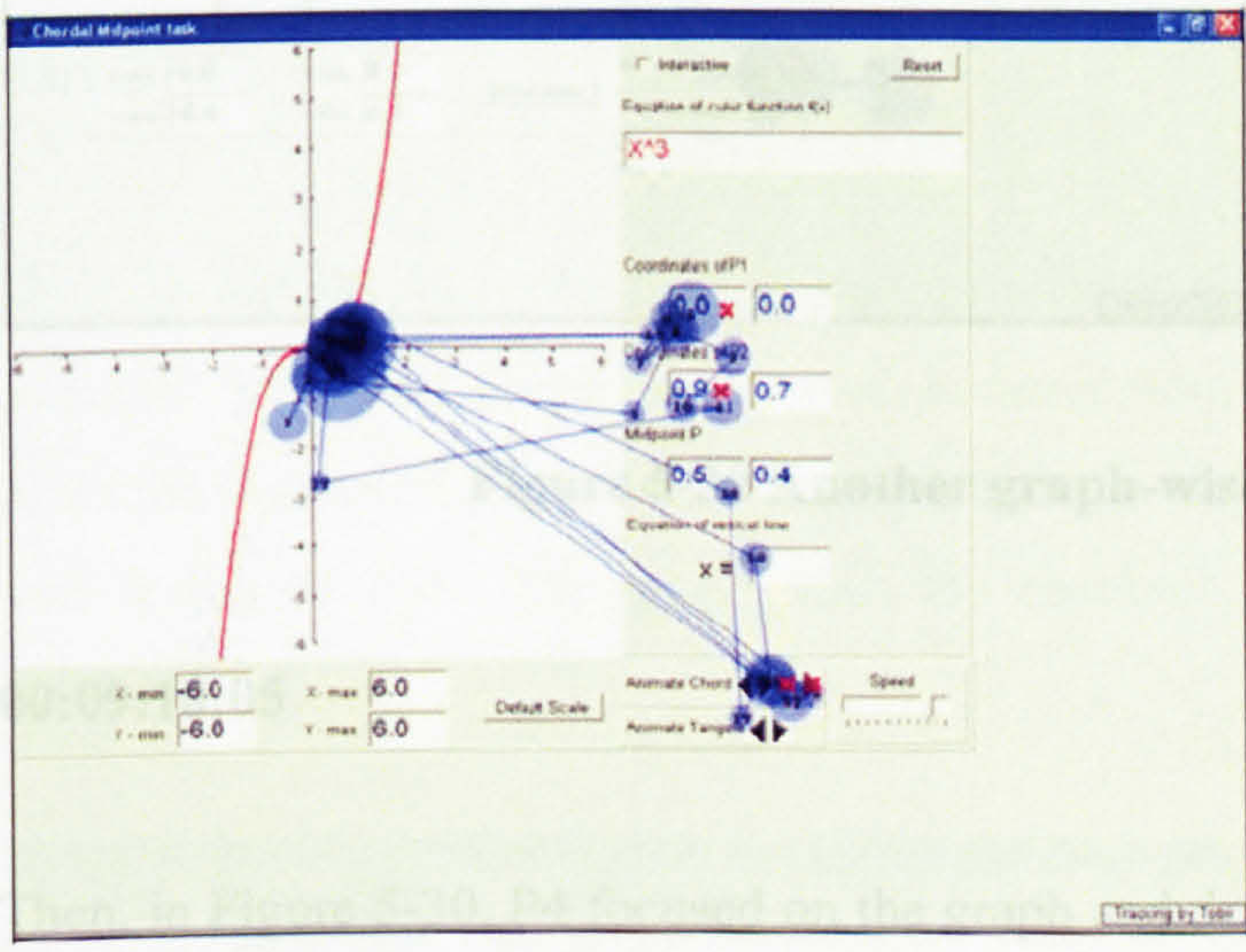
(P4 looked at the Tablet PC. Then, she glanced back at the interface)
(00:08:32:20)
"Let's assume something... If I have line from here to here and moving this way it goes asymptotic... asymptotic... If I have it this way zero zero to here it goes this way it is also asymptotic..."

Figure 5-27 A representation of P4's glanced at the interface of P4



Code: visual (re-representing)
(P4 sketched on the Tablet PC)
(00:08:32:20)
(left part of the worksheet) A cubic graph with an arrow pointing towards the left. Wrote "asymptotic"
(Right part) A cubic with an arrow drawn going to the right. Wrote "asymptotic"

Figure 5-28 A visual re-representing strategy of P4



Code: mouse-drawing (imagining)
(P4 looked back at the interface. She pointed at (0, 0) and moved the mouse pointer downward)
(00:08:55:10)
"let me try something... zero zero and... it move just this way... let's move down"

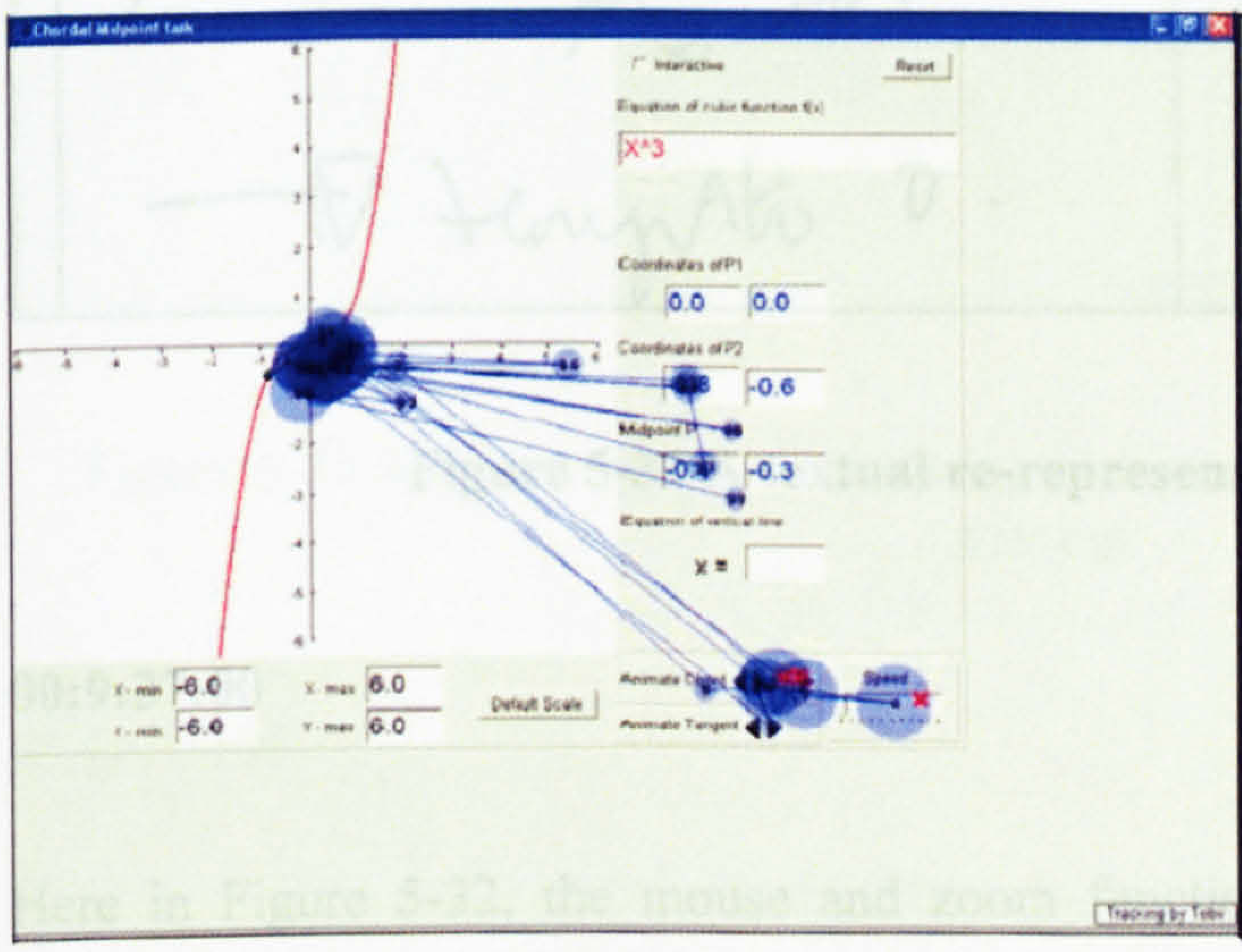
Figure 5-29 Another mouse-drawing strategy of P4

00:08:32:20

P4 recorded her guess drawing on the Tablet PC (Figure 5-28). Whilst sketching, she also glanced back at the interface and talked (Figure 5-27). She used some arrow diagrams in her sketches. The diagrams are representations not found in the interface. For this reason, the event was coded as a *visual re-representing strategy.*

00:08:55:10

P4 pointed at the graphical point (0, 0). Point (0, 0) was mentioned in her talk (Figure 5-29). She moved the mouse pointer upward depicting some lines being drawn from point (0, 0). This was coded as a *mouse-drawing strategy*.



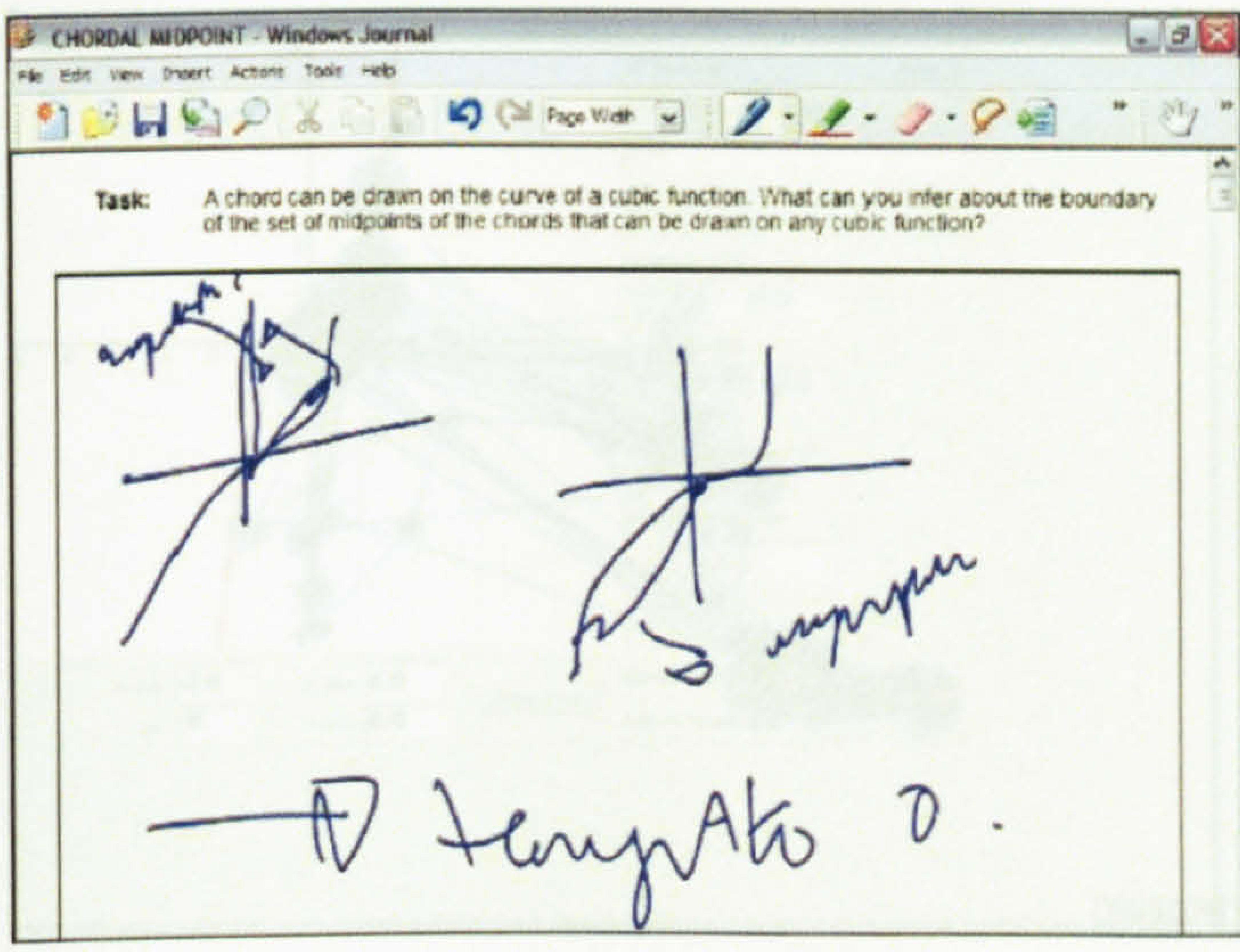
	Code: graph-wise (representation-specific) *(P4 animated the graph and watched it)* *(00:09:10:05)* *"So all it does... So all it does... comes down comes down... until it becomes tangent line zero and it changes direction... yeah..."*

Figure 5-30 Another graph-wise strategy of P4

00:09:10:05

Then, in Figure 5-30, P4 focused on the graph and described the graph's behaviour. Thus, the code is a *graph-wise strategy*.

Based on Figure 5-30 above, P4 tried to represent her talk on the Tablet PC. She wrote the word "tangent" (Figure 5-31 below). This was coded as *textual re-representing strategy*.

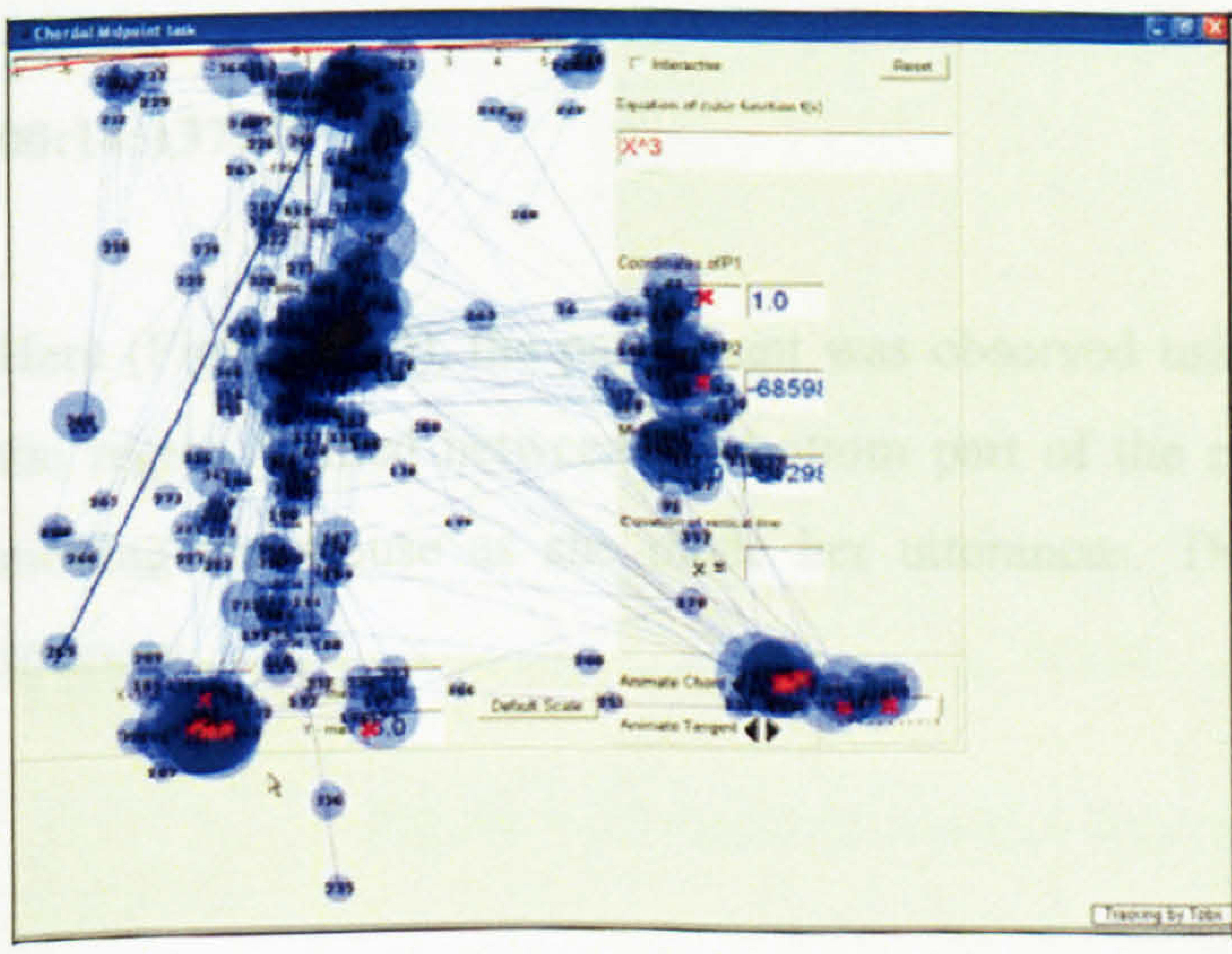


Code: textual (re-representing)
(P4 looked at the Tablet PC. She wrote on the Tablet PC.
(00:09:10:05)
Drew an arrow pointing left and wrote "tangent 0")
"Alright... Ok... and then if I used one to minus one... (Inaudible) (So it just moves along) Ah! Ok...using minus one to one... It goes down that way... So it crosses the line... But... it cuts one zero..."

Figure 5-31 A textual re-representing strategy of P4

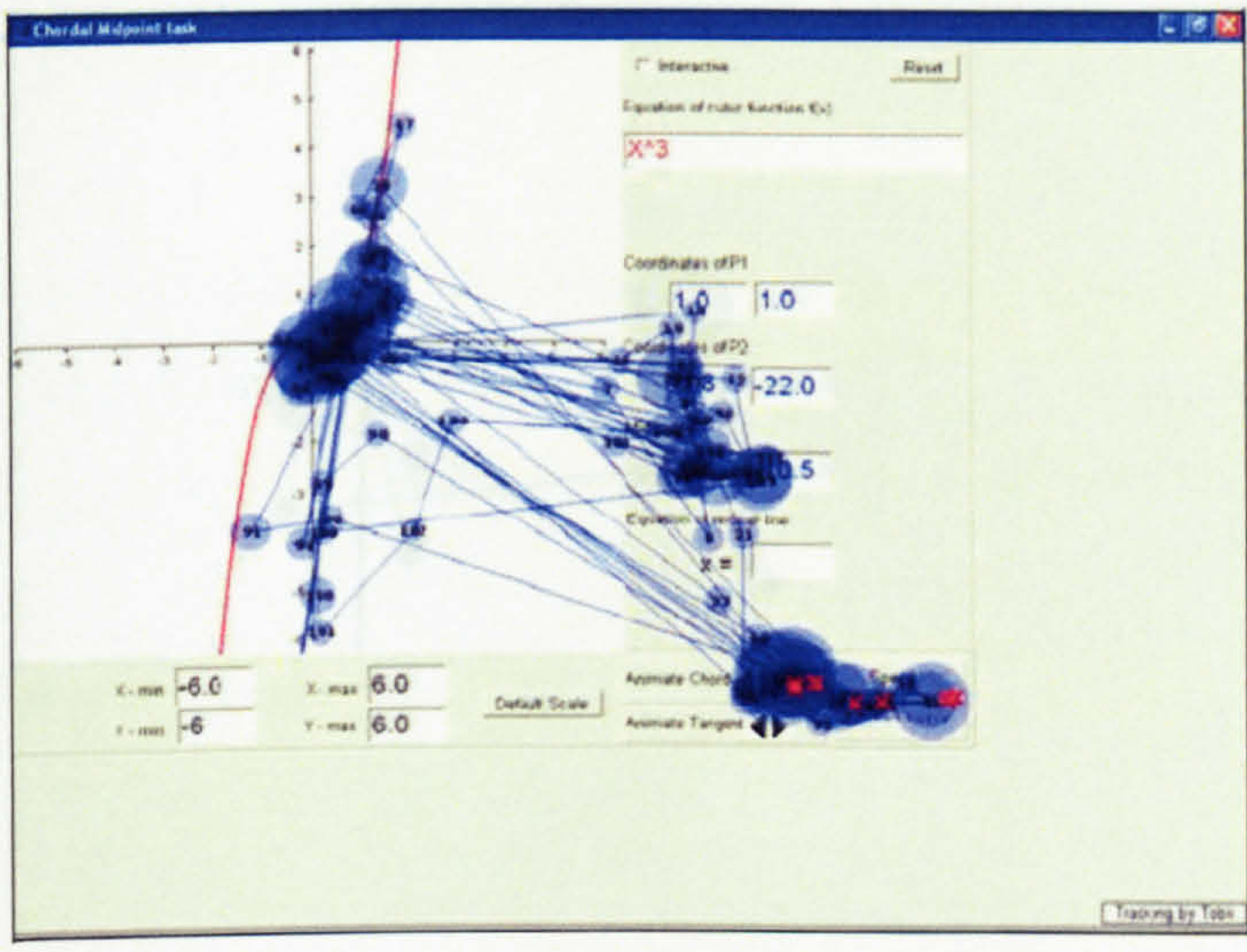
00:9:37:00

Here in Figure 5-32, the mouse and zoom function operations were in unsystematic manner. This was coded as a *random-interaction strategy*. However, the eye-marks direction was alternating from the graph and the numbers. Her talk also showed that she was trying to relate the graph with the numbers. Also coded as a *graphic-numeric strategy.*



Code: graphic-numeric; random-interaction
She animated the graph in different directions. She looked at the graph and numbers back and forth as she watched the graph. She changed the zoom scale
(00:09:37:00)
"So that's down right...and it still keeps going...How long can this go to?... I'm still trying to figure out what's going on...That keeps going... that just never...When does the midpoint end..."

Figure 5-32 A graphic-numeric and random-interaction strategy of P4



Code: trial-and-error; random-interaction; graphic-numeric
P4 gasped. She animated the graph in every direction. She changed the speed of animation. She looked at the graph and numbers whilst watching the animation of the graph.
(00:13:14:00)
"It gets to... who care... minus six back to that... tangent is here... it goes up... so obviously... it's doing that... let's stop there... The midpoint is minus zero point nine minus ten point five..."

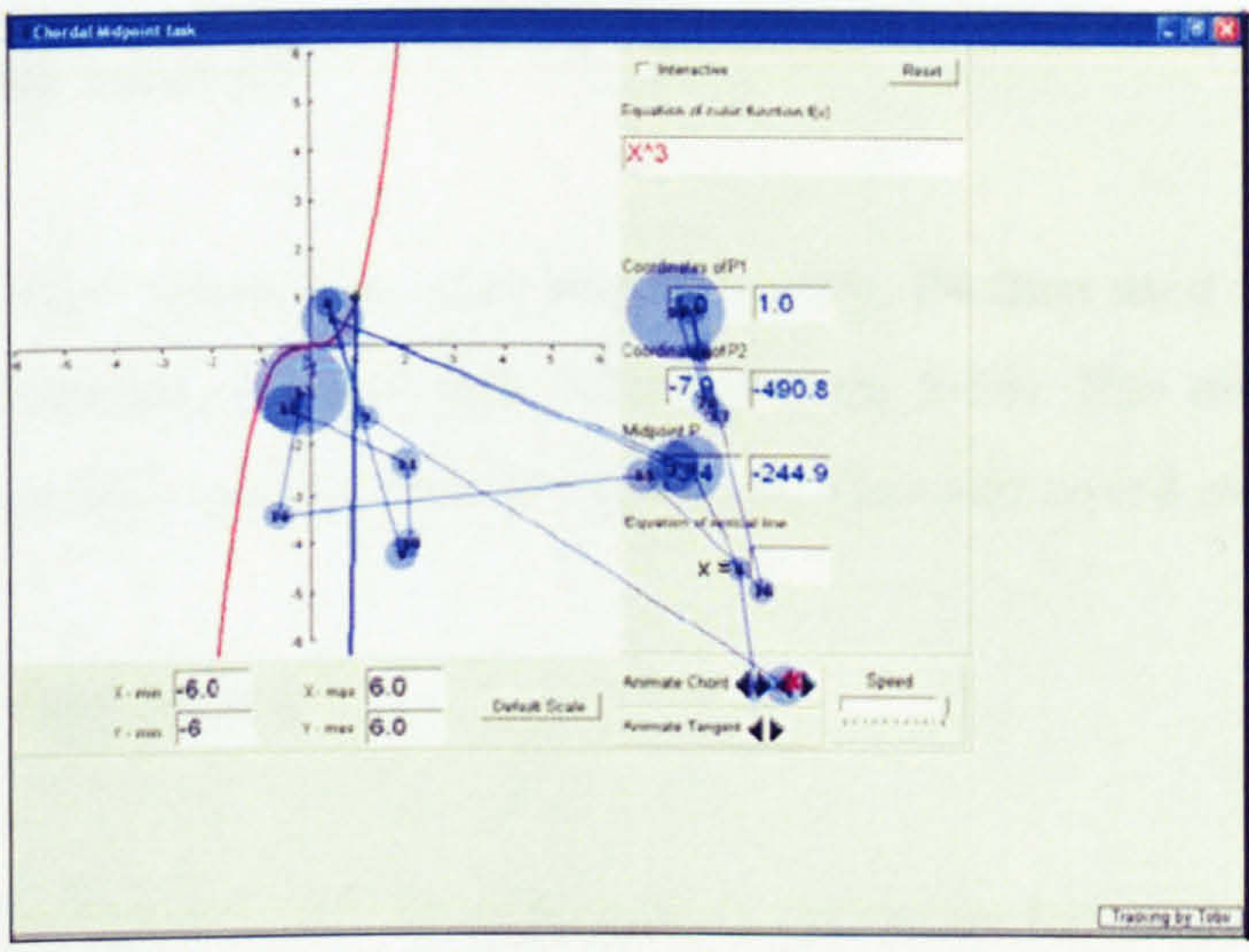
Figure 5-33 Another graphic-numeric, trial-and-error and random-interaction strategy

00:13:14:00

The same mouse operations, as in Figure 5-29, were employed here (Figure 5-33). However, the talk made it apparent that random actions were being tested. This event was coded as *trial-and-error* and *random-interaction* strategies. Also, the talk with the eye-marks (similar to the explanation for Figure 5-32) provided the same reason to code this event as a *graphic-numeric strategy.*

00:14:137:22

Here (Figure 5-34), the participant was observed using the mouse pointer moving along the region formed between the bottom part of the cubic graph and the chord. She was moving the mouse as she made her utterances. This was coded as a *mouse-drawing strategy.*

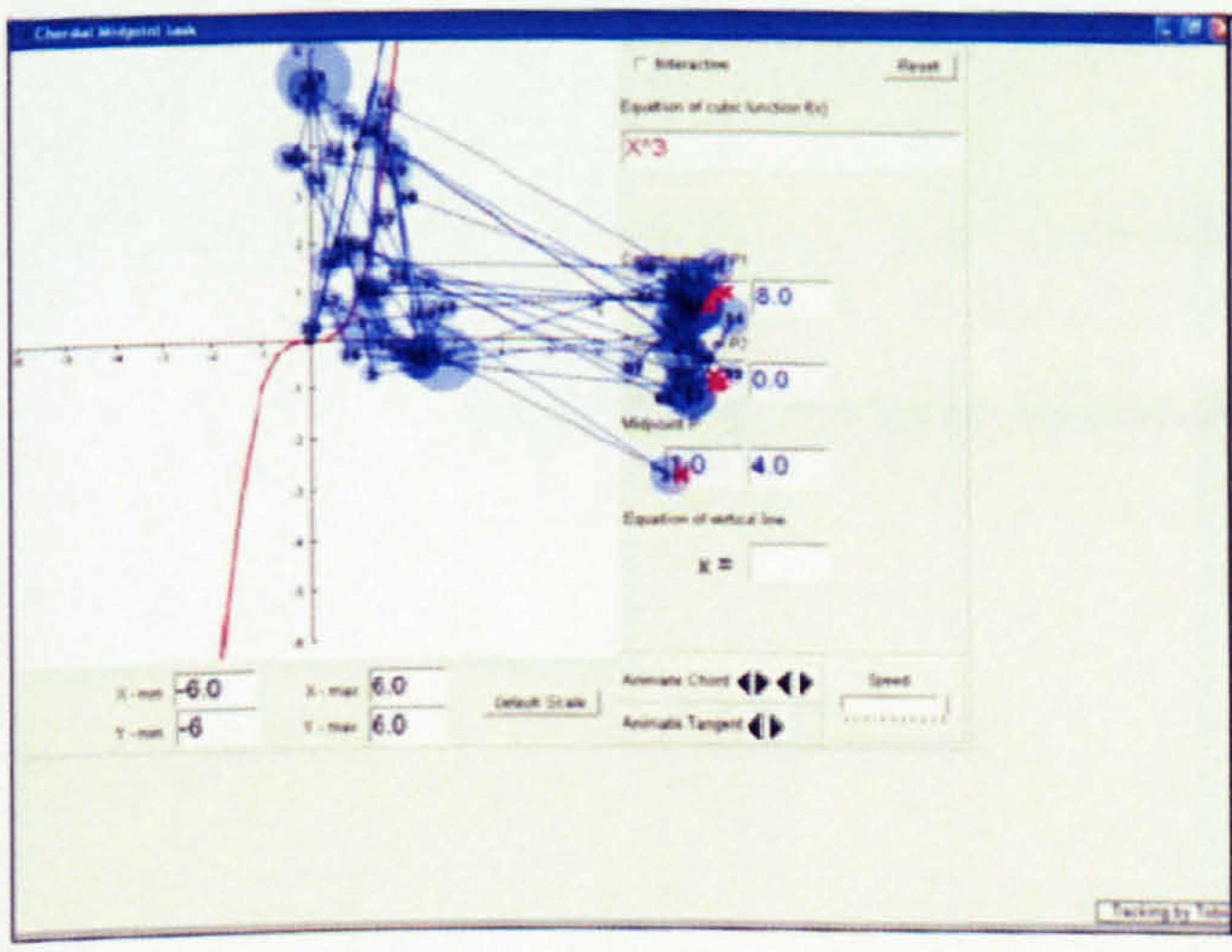


Code: mouse-drawing (imagining)
(P4 then stopped the animation then pointed the mouse at the region inside the curve and the graphical line. She moved the mouse pointer)
(00:14:13:22)
"It will never ever comes cross... Something... it never comes across..."

Figure 5-34 Another mouse-drawing strategy of P4

00:14:27:18

P4 used a different *random-interaction strategy* is this particular event. She inputted different numeric entry at random without animating them. The numbers entered did not follow any order as shown in the talk (Figure 5-35).



Code: random-interaction
(P4 inputted different numbers. She did not animate the graph rather she just watched the graph)
(00:14:27:18)
"Ok let us try something else... three zero... In fact, two... It is doing it again... one point five..."

Figure 5-35 Another random-interaction strategy of P4

00:15:01:03

After entering another numeric entry, P4 then used the mouse. P4 pointed at the number mentioned in her talk below (Figure 5-36). She moved the mouse pointer left to right depicting some lines being drawn. This was coded as a *mouse-drawing strategy.*

00:15:47:10

In Figure 5-37, P4 attempted to generalise her conjecture based on what she had written. This was not part of the strategy scheme; therefore, no code was given to this action.

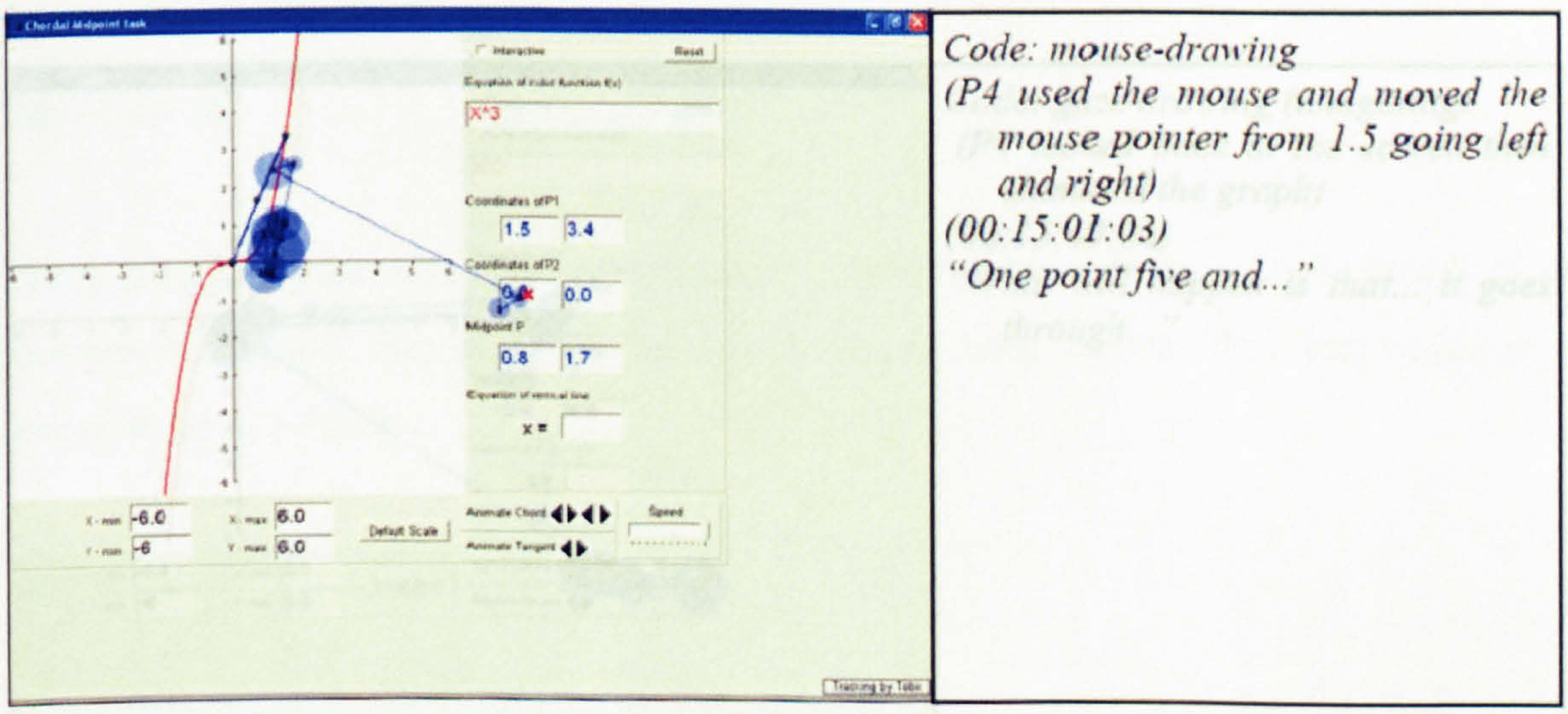
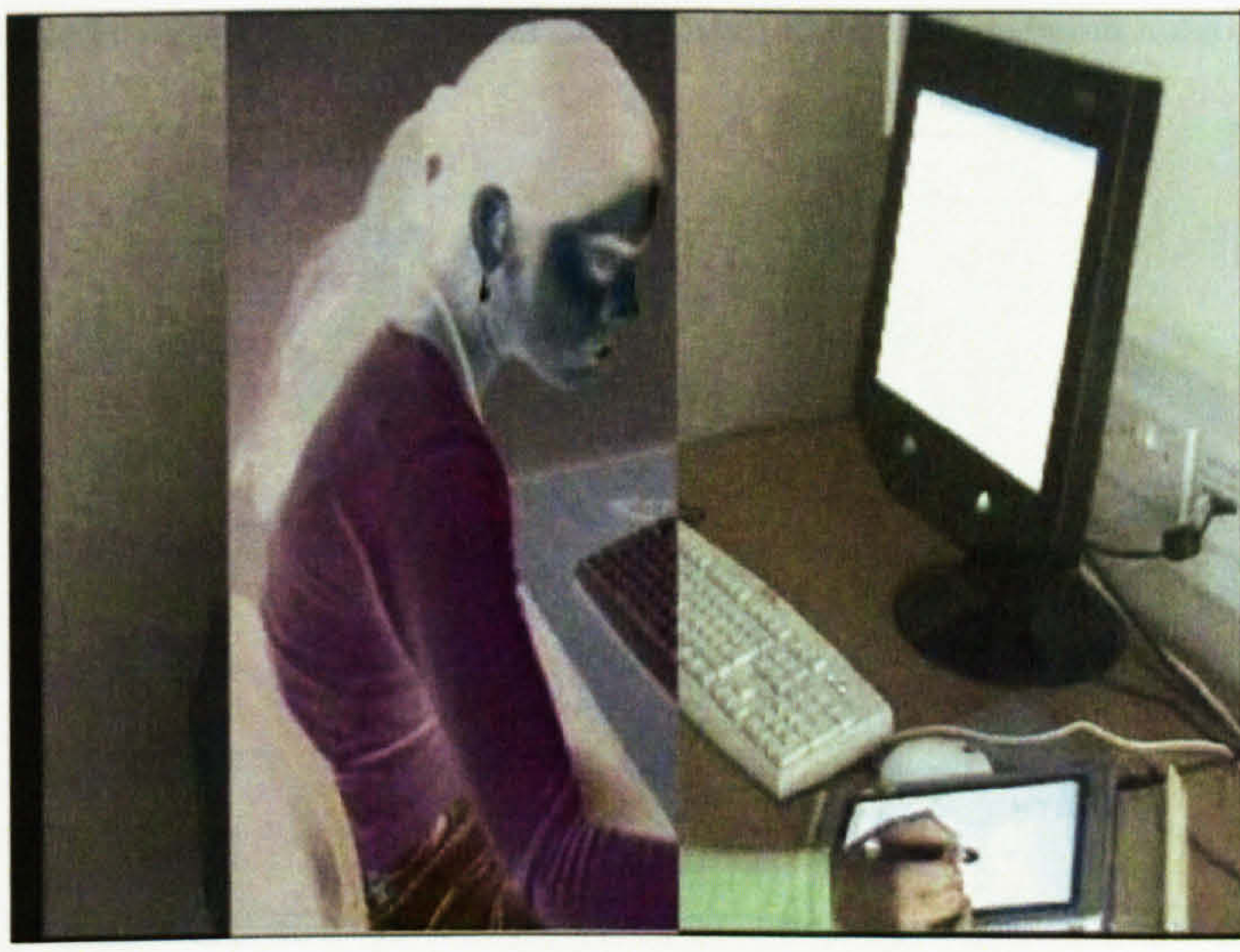
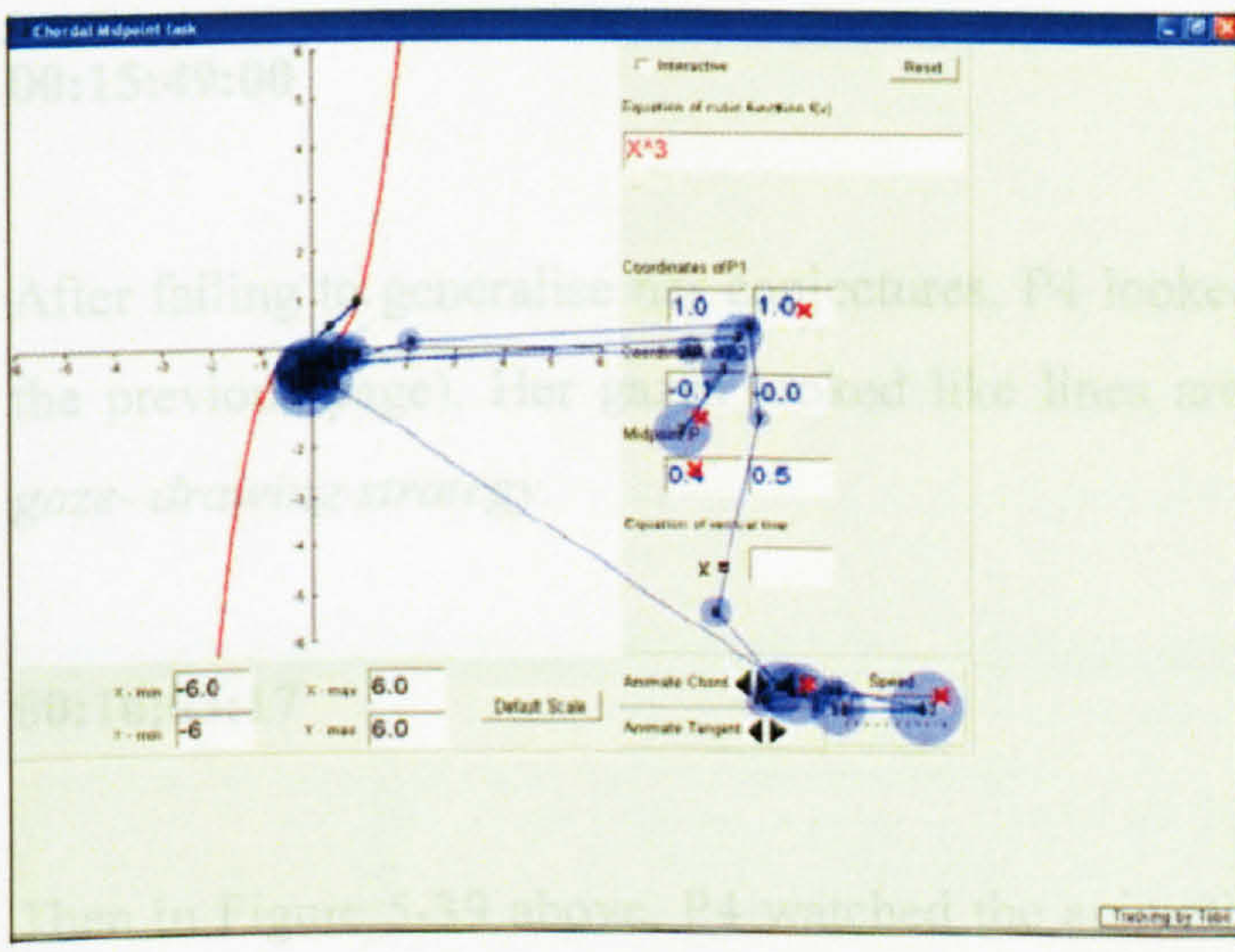


Figure 5-36 Another mouse-drawing strategy of P4



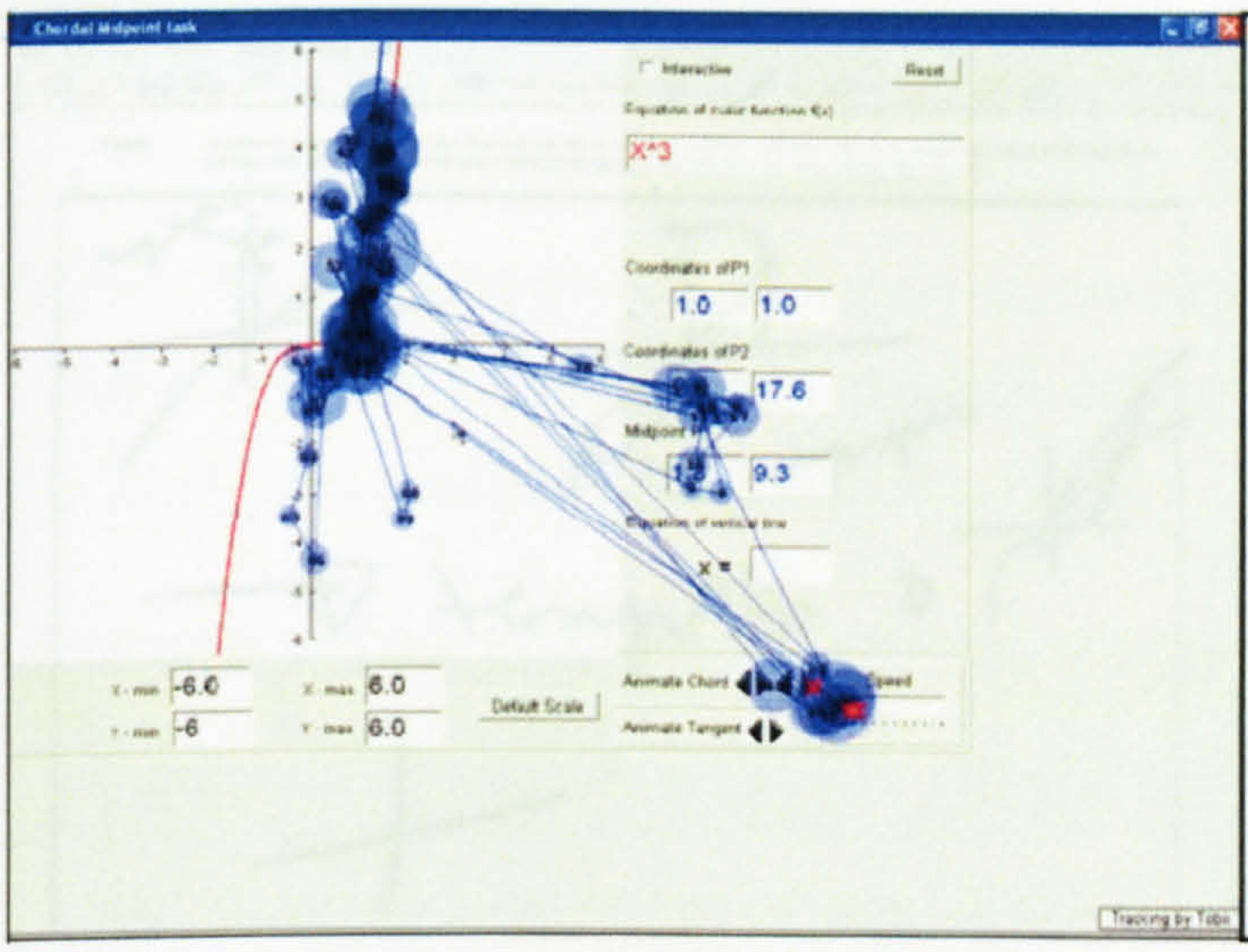
(P4 looked at the Tablet PC)
(00:15:47:10)
"OK... What conclusion can I make from this... I know what's happening when the tangent is zero... when the tangent is zero... All you have is that the midpoint keeps going this way until it's asymptotic to that axis... And then we have the next one... it is asymptotic to that axis... when we do... but then it passes through that point... and then the same thing happens here..."

Figure 5-37 P4 writing her conjecture



Code: gaze-drawing (imagining)
(P4 looked back at the screen then animated the graph)
(00:15:49:00)
"What will happen is that... it goes through..."

Figure 5-38 Another gaze-drawing strategy of P4



Code: guess-and-test; graph-wise
(P4 continued animating and watching the graph. She looked at the graph and occasionally looked at the numbers)
(00:16:03:17)
"It goes to zero zero... It then goes through... It becomes a tangent to it... It shifted... it shifts... and then it becomes asymptotic to something which I can't figure... but it is... Is it the point it self one zero... It could be the point itself one zero... because it never ever comes across"

Figure 5-39 Another guess-and-test and graph-wise strategy of P4

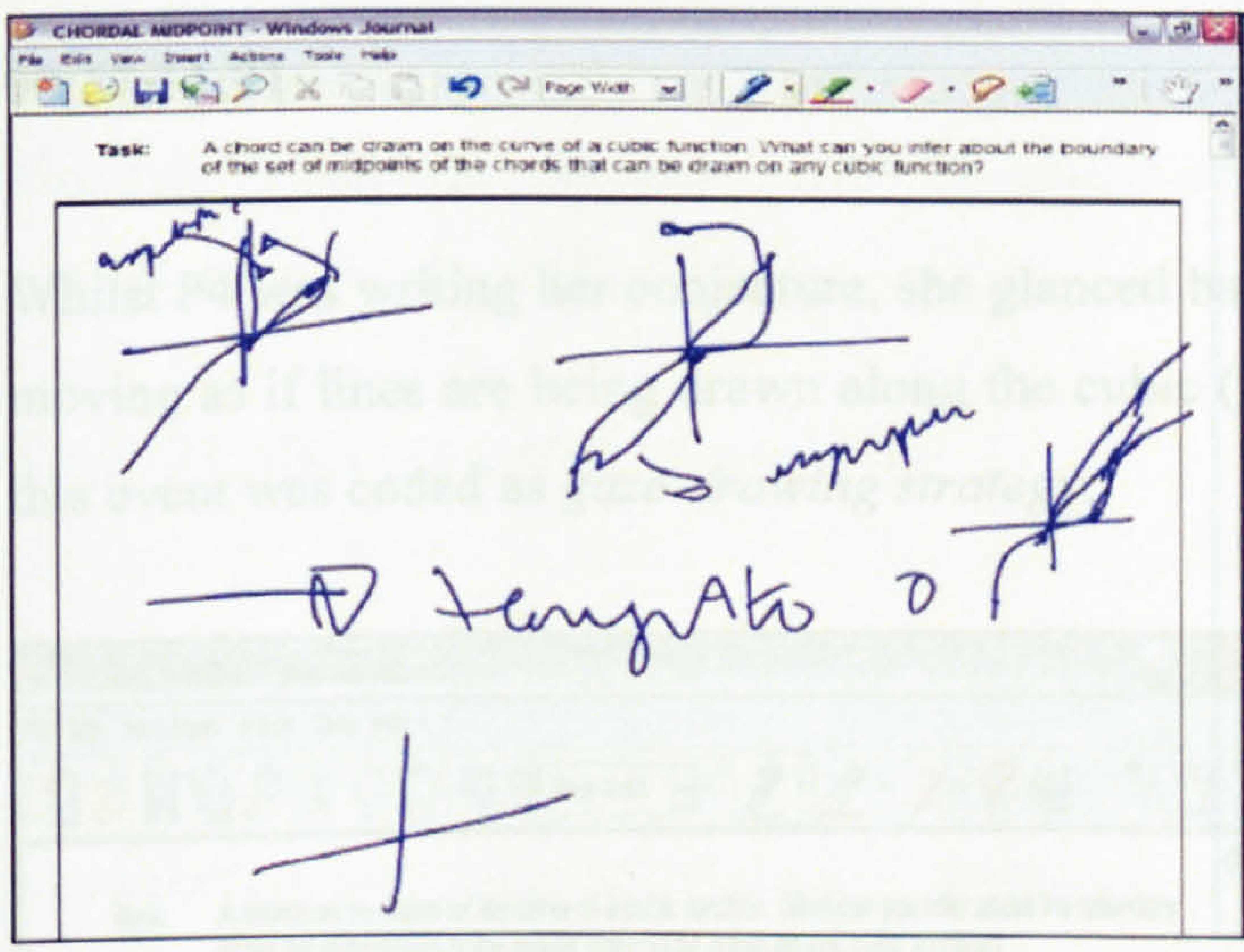
00:15:49:00

After failing to generalise her conjectures, P4 looked at the screen again (Figure 5-38 of the previous page). Her gazes looked like lines are being drawn. This was coded as a *gaze- drawing strategy.*

00:16:03:17

Then in Figure 5-39 above, P4 watched the animation of the graph. The utterance made was about the graph's behaviour; thus, a *graph-wise strategy*. Then she arrived at another guess as shown in her talk. This was also coded as a *guess-and-test strategy.*

P4 used the Tablet PC. She drew some lines to test her guess (in Figure 5-39).

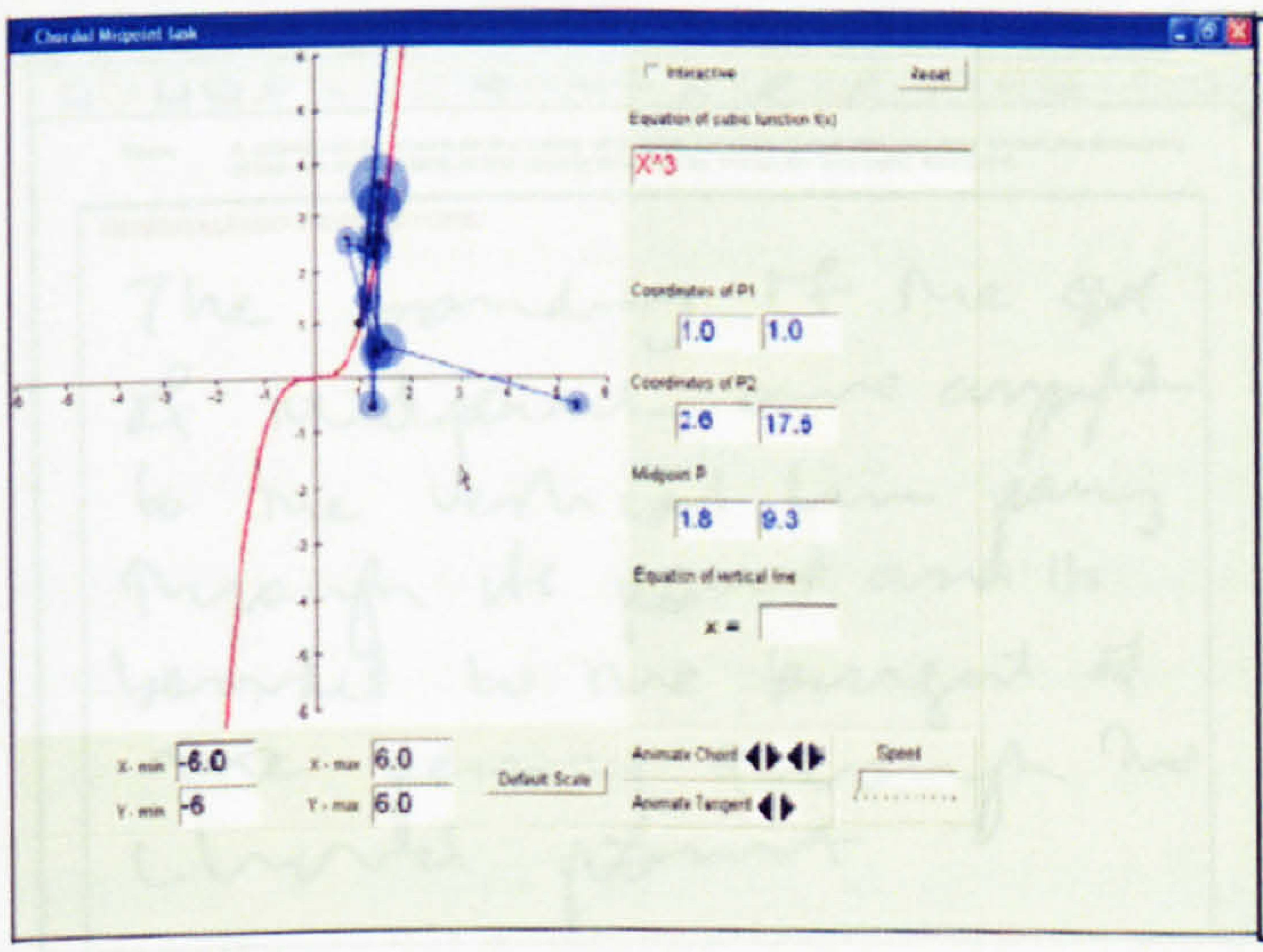


Code: pen-drawing
(P4 wrote at the Tablet PC. Pen pointer moved on XY plane)
(00:16:48:05)
(Right-most part of the Tablet PC) Drawing line from (0, 0) going upwards.
(Bottom part of the Tablet PC) Drew an XY plane. Mouse pointer making some lines.

Figure 5-40 A pen-drawing strategy of P4

00:16:48:05

In Figure 5-40 (bottom part of the Tablet), the 'pen pointer' (similar to that of the mouse pointer for desktop) was seen making lines along a blank XY plane. This was coded as a *pen-drawing strategy.*

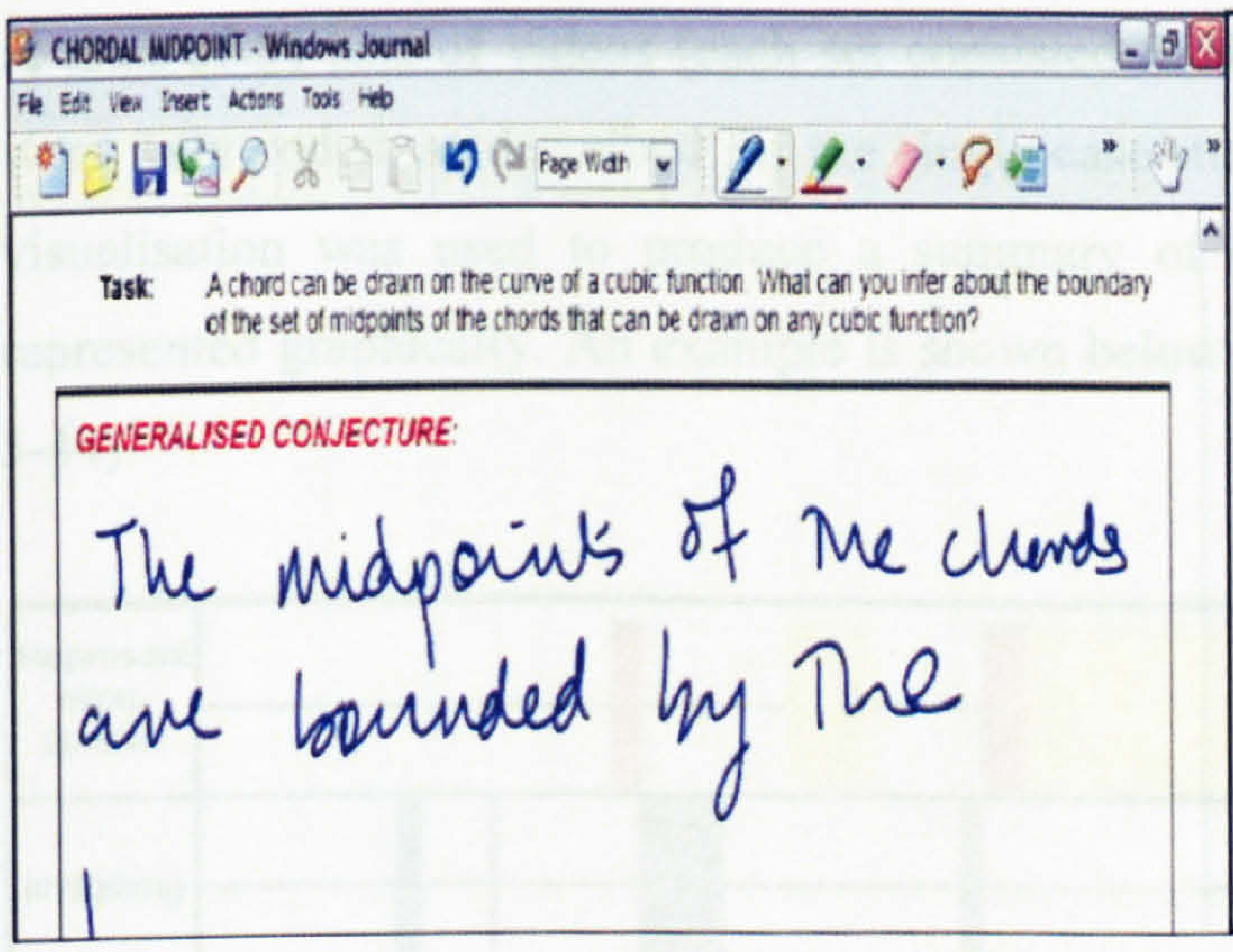


Code: gaze-drawing
(P4 glanced back on the computer screen)
(00:17:11:01)

Figure 5-41 Another gaze-drawing strategy of P4

00:17:11:01

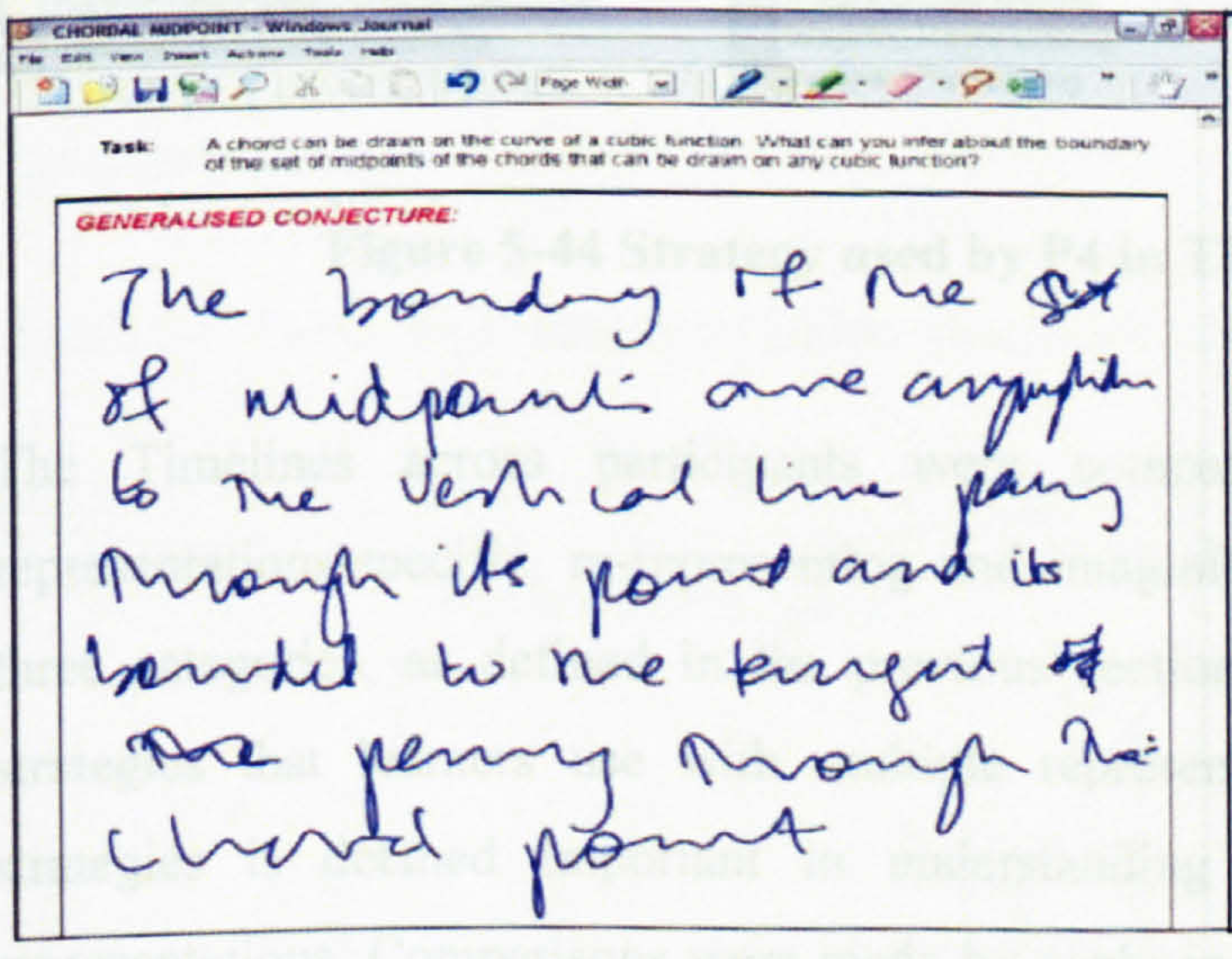
Whilst P4 was writing her conjecture, she glanced back on the Tablet PC. The gazes were moving as if lines are being drawn along the cubic (Figure 5-41). This is the reason why this event was coded as *gaze-drawing strategy*.



No code given

(P4 wrote. She erased what she had written)
(00:18:50:05)
"The chords... the midpoints... of the chord... are bounded... No. wait I need to check something... I did zero one... I did one not... It goes... blah blah blah blah... Ok yeah it seems... are bounded by the bounded by the points... what will I say... by the... Noo."

Figure 5-42 P4's conjecture that was overwritten



(P4 wrote on the Tablet PC. She was verbalising what she had written)
(00:18:50:05)
"Uhm... I'll make a conclusion I know it's wrong Alright... let's see what's the inference from this... The chords... the midpoints... of the chord... are bounded... No. wait I need to check something... I did zero one... I did one not... It goes... blah blah blah blah... Ok yeah it seems... are bounded by the bounded by the points... what will I say... by the... Noo."

Figure 5-43 P4's final conjecture

She tried to make her conclusion, speaking aloud what she was writing on the Tablet PC. She then erased what she had written (Figure 5-42). P4 then decided to generalise her conjecture as shown in Figure 5-43.

5.4.3 'QUANTITATIVE' ANALYSIS

A total of 54 sets of videos (each set consisted of the gaze, action and writing videos) were fully coded, as described for the single case study in section 5.4.2 above. Timeline visualisation was used to produce a summary of data in which observation can be represented graphically. An example is shown below representing P4's strategies (Figure 5-44).

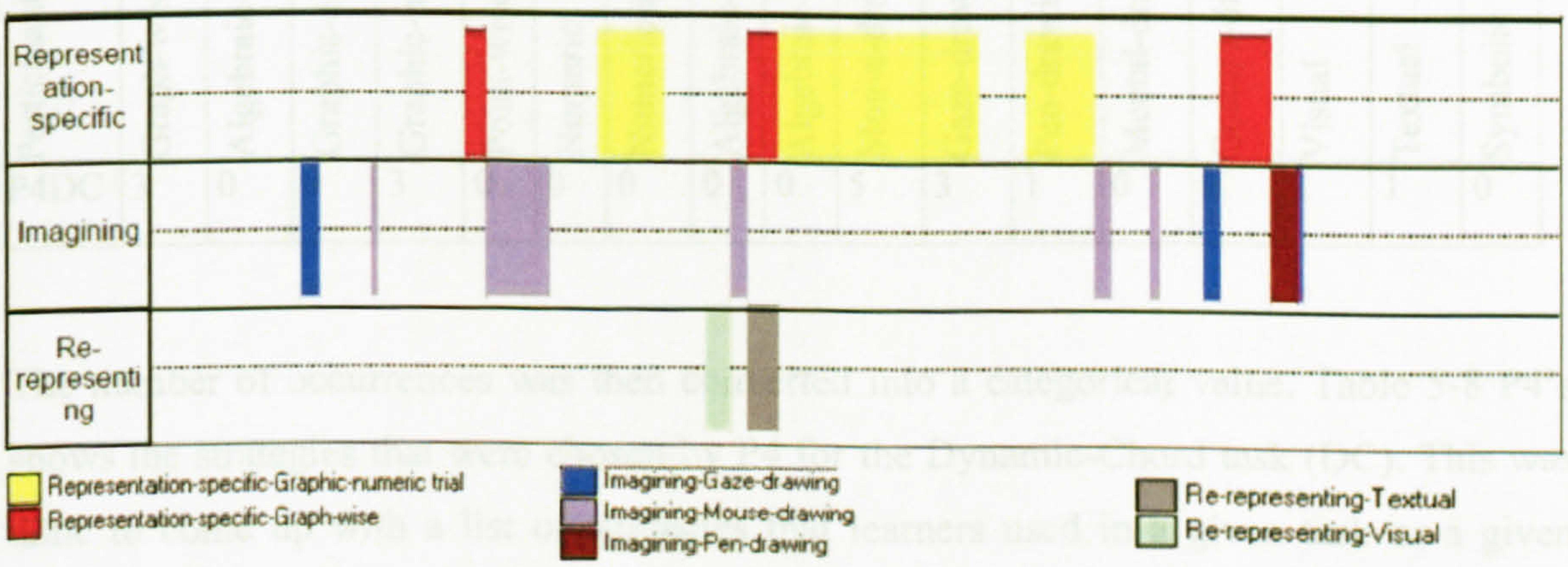


Figure 5-44 Strategy used by P4 in Timeline visualisation

The Timelines across participants were compared by strategy categories (i.e. representations-specific, re-representing and imagining). The characterisations of these three categories, as defined in the previous section 5.4.1, are directly related to the strategies that learners use with multiple representations. The comparison of these strategies is deemed important in understanding what learners do with multiple representations. Comparisons were made by analysing the Timelines for the participants who did the tasks without varying instantiations to those who did the task with varying instantiations.

A summary of the occurrence of each strategy employed by each of the participants was generated automatically using INTERACT. Table 5-7 show the number of times a certain strategy was used by P4. This data provide the kinds of representations a learner focus onto in completing a task.

Table 5-7 Number of occurrence of strategies used

Code	Strategies																	
	Representation-specific									Imagining					Re-representing		
Participant Task	Graph-wise	Algebraic-chunking	Graphic-algebraic	Graphic-numeric	Point-wise	Numeric	Numeric-algebraic	Algebraic-graphic	Algebraic-manipulation	Mouse-drawing	Gaze-drawing	Pen-drawing	Mental-drawing	Gesture-drawing	Visual	Textual	Symbolic
P4DC	3	0	0	3	0	0	0	0	0	5	3	1	0	0	1	1	0

The number of occurrences was then converted into a categorical value. Table 5-8 P4's shows the strategies that were chosen by P4 for the Dynamic-Chord task (DC). This was done to come up with a list of strategies that learners used in a given task or a given instantiation.

Table 5-8 Table representation of the strategies used by P4

Code	Strategies																	
	Representation-specific									Imagining					Re-representing		
Participant Task	Graph-wise	Algebraic-chunking	Graphic-algebraic	Graphic-numeric	Point-wise	Numeric	Numeric-algebraic	Algebraic-graphic	Algebraic-manipulation	Mouse-drawing	Gaze-drawing	Pen-drawing	Mental-drawing	Gesture-drawing	Visual	Textual	Symbolic
P4DC	✓			✓						✓	✓	✓			✓	✓	

The list of strategies (Table 5-8) can then be transformed into a panelled column chart using SPSS™. The number of participants who chose the same strategies can be easily shown, by task or by instantiation. In Figure 5-45, for example, *representation-specific* strategies chosen by 18 participants of this study for the chord task are shown. By inspection, one can see that the *graph-wise* strategy was used by all the participants. However, no algebraic-related *representation-specific* strategies were used under this task.

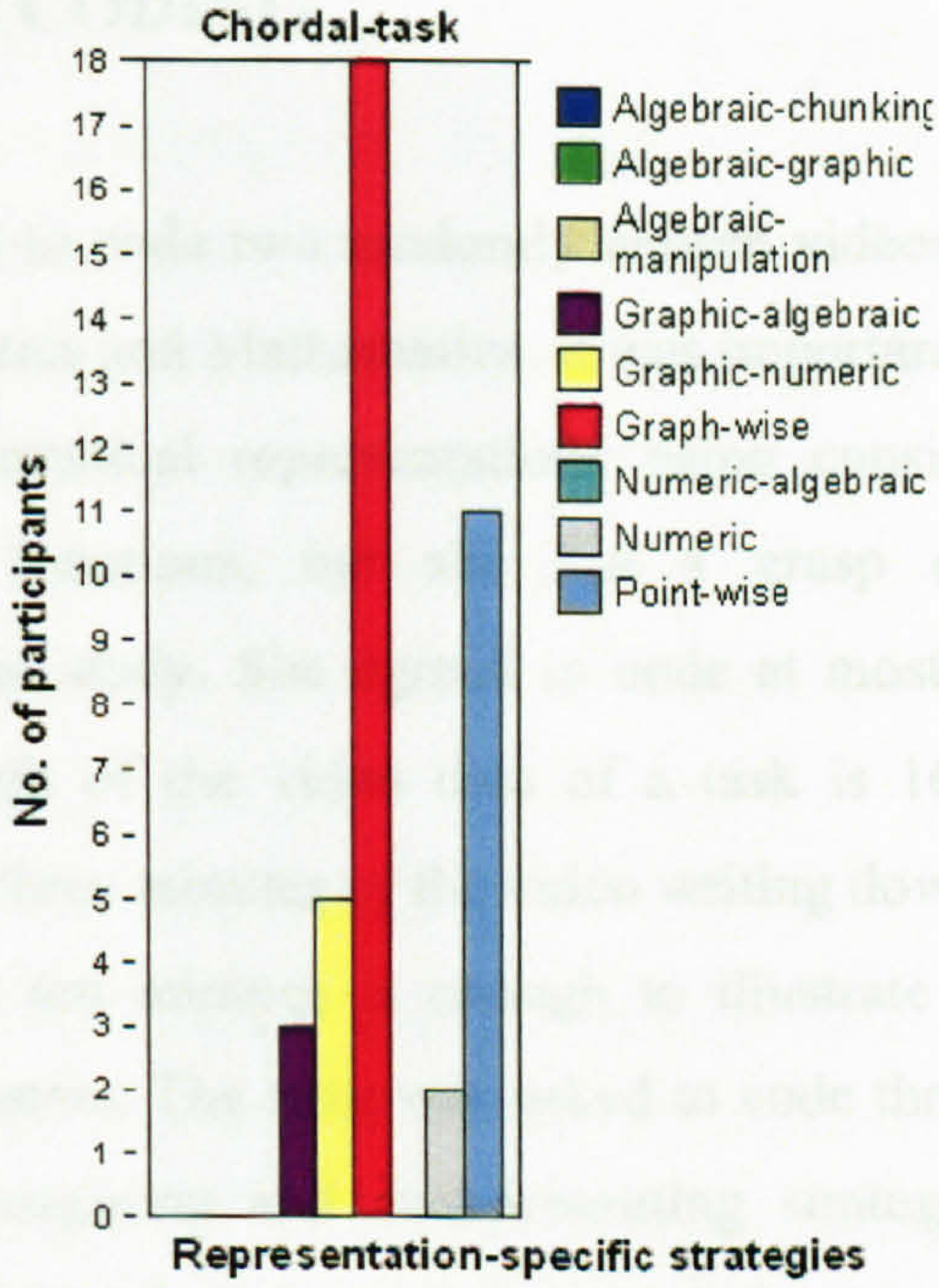


Figure 5-45 Representation-specific strategies used in chord task

5.4.4 LINKING RESULTS TO VIDEOS

The quantitative data have allowed for investigating obvious patterns as can be detected from the Timeline and the panelled column chart. For example, the strategies that were used and not used can be easily seen. The results of the quantitative analysis were then linked back with the videos. Taking P4's data, similarities between the graph-wise strategies used across all the tasks were re-watched and interpreted. The strategies that occurred only in one task and not in other tasks were also re-watched looking for possible reasons that could explain the differences in the strategy used.

5.4.5 INTERPRETING RESULTS

The interpretations were drawn from the typicality or atypicality of the strategies employed; and its relation to the nature of the task or the instantiation. Other possible explanations that can be related to the results were also considered, such as the researcher's presence, set-up, and the like. Furthermore, reflections were made on the results with pertinent previous claims identified in the literature review chapter.

5.5 INTER-RATER CODING

Another 'rater' was asked to code two randomly chosen videos. The rater is female and has a background in Statistics and Mathematics. It was important that the rater has enough background in the mathematical representations being considered. The rater has no experience in teaching functions, but she has a grasp of the concepts of the representations used in the study. She agreed to code at most 10 minutes each of two videos. The average length of the video data of a task is 16 minutes. Typically, the participants spent the last three minutes of the video writing down their final answers. It is reasonable to accept that ten minutes is enough to illustrate the consistency between codings of two different raters. The rater was asked to code three types of strategies (i.e. representation-specific, imagining and re-representing strategies). The other types of strategies (i.e. generic and interface-supported strategies) were also given to the rater and were explained. The rater only agreed to spend two hours each session. Therefore, she was not asked to code the generic and interface-supported strategies. There were three sessions which entailed about six and a half hours of the rater's time.

5.5.1 PROCEDURE

Preliminary session. First, each strategy in the coding scheme was explained one by one. The rater asked questions about each of the strategies. For example, there was a question asked about the distinction between a graphic-algebraic and an algebraic-graphic strategy; and this was illustrated. After all the questions about each of the strategies were made clear, the rater was taught the basic operations of coding with INTERACT. The author showed a coded trial task video to the rater which was played event by event. An "event" as defined in section 5.2.2 is a part of the video that is given a specific code. Then, another video was shown, this time replaying the event without the rater seeing the assigned code. After replaying each event, the rater was asked what code she would assign to the event being shown. For every code given by the rater that matched the author's coding, the rater was asked to explain why. When the explanation did not fit the characterisation of the strategy code, the author again clarified the description of that particular strategy. When the rater's code did not match the author's code, an agreement was reached. In those instances, the rater was found to get confused with coding the strategies and the possible main reason for this was that the video was only seen for the first time. This was taken as an opportunity to emphasise to the rater about the importance of the phases of analysis considered in section 5.4. A video was given to the rater to familiarise herself with. She agreed to code the video after three days from the first preliminary session. She also agreed to code another video on another date. Therefore there were two videos that this rater coded. The videos were coded first, and then followed by an interview.

Coding session. The first video coded was an 8-minute video of 'the tangent task in an 'Interactive instantiation' by P15. It took the rater 30 minutes coding the video category by category (representation-specific, imagining and re-representing strategies). Questions about the reason for coding the video were asked immediately after coding. The author took note of reasons given that he thought did not match the strategy characterisation. After coding, the author showed his coding and again an agreement was reach for non-matching codes.

The rater agreed to code a 10-minute excerpt of a 25-minute video. The task was 'chord task in Static instantiation' by P1. This session took place week after coding the first video. The video was also given at this session due to some irregularities with schedule and technical constraints. Therefore the rater had to first start familiarising herself with the video. The coding took almost two hours. The coding was discussed immediately afterwards, similar to the procedures above.

5.5.2 Results and discussions of the inter-rater coding

The results presented here are about the two videos coded. In the two videos coded, the start and end time (timecodes) of some of the events coded did not match exactly (see Table 5-9 for the coded events with timecodes). Getting the timecodes to exactly match is impossible to achieve because difference of timecodes in coding could range from 1 frame per second to a minute. For example, the rater was not expected to go through each of the coded events to change each start and end time to match the start and end of an utterance for a particular strategy. However, the difference between the timecodes is within the utterance where a strategy was identifiably changing.

First video. Figure 5-46 below, shows three pairs of Timelines where the first pair refers to representation-specific strategies coded, followed by the imagining, then by the re-representing strategies.

Some of the events coded did not match and some matched (Figure 5-46). There were some events coded by the rater that did not appear in the author's coding. And also, some events coded by the author did not appear in the rater's coding.

Table 5-9 Inter-rater coding: examples of non-matching time codes

Entry	Exit	Representation-specific	Imagining	Re-representing
P1 SC (author's rating)				
00:00:30:04	00:00:50:08	Graphic-numeric		
00:00:50:08	00:01:50:16			
00:01:50:16	00:03:09:07			
00:03:09:07	00:04:27:02			
00:04:27:02	00:04:34:14			
00:04:34:14	00:05:07:22			
00:05:07:22	00:05:26:20			
00:05:26:20	00:06:04:11		Mouse-drawing	
00:06:04:12	00:07:14:24		Gaze-drawing	
00:07:15:01	00:08:44:13		Gaze-drawing	
00:08:44:13	00:09:12:23	Point-wise		
00:09:12:23	00:09:14:10		Gaze-drawing	
00:09:14:11	00:10:02:06	Point-wise	Gaze-drawing	
00:10:02:06	00:11:25:13	Numeric-trial	Mouse-drawing	
00:11:25:14	00:12:00:17	Graph-wise	Gaze-drawing	
P1 SC (Researcher's rating)				
00:00:33:02	00:00:45:20	Graphic-numeric		
00:03:05:04	00:03:50:04	Graph-wise		
00:04:42:12	00:05:22:15	Graph-wise		
00:05:36:20	00:06:05:06		Mouse-drawing	
00:06:55:07	00:07:43:03		Mouse-drawing	
00:08:12:05	00:08:25:21		Gaze-drawing	
00:08:32:08	00:09:30:24	Graph-wise		
00:10:09:21	00:11:00:03	Graph-wise		
00:09:36:16	00:09:43:24		Gaze-drawing	
00:11:02:20	00:11:24:22	Graph-wise		
00:11:22:04	00:11:51:15		Gaze-drawing	
00:11:34:07	00:11:51:15	Graph-wise	Gaze-drawing	

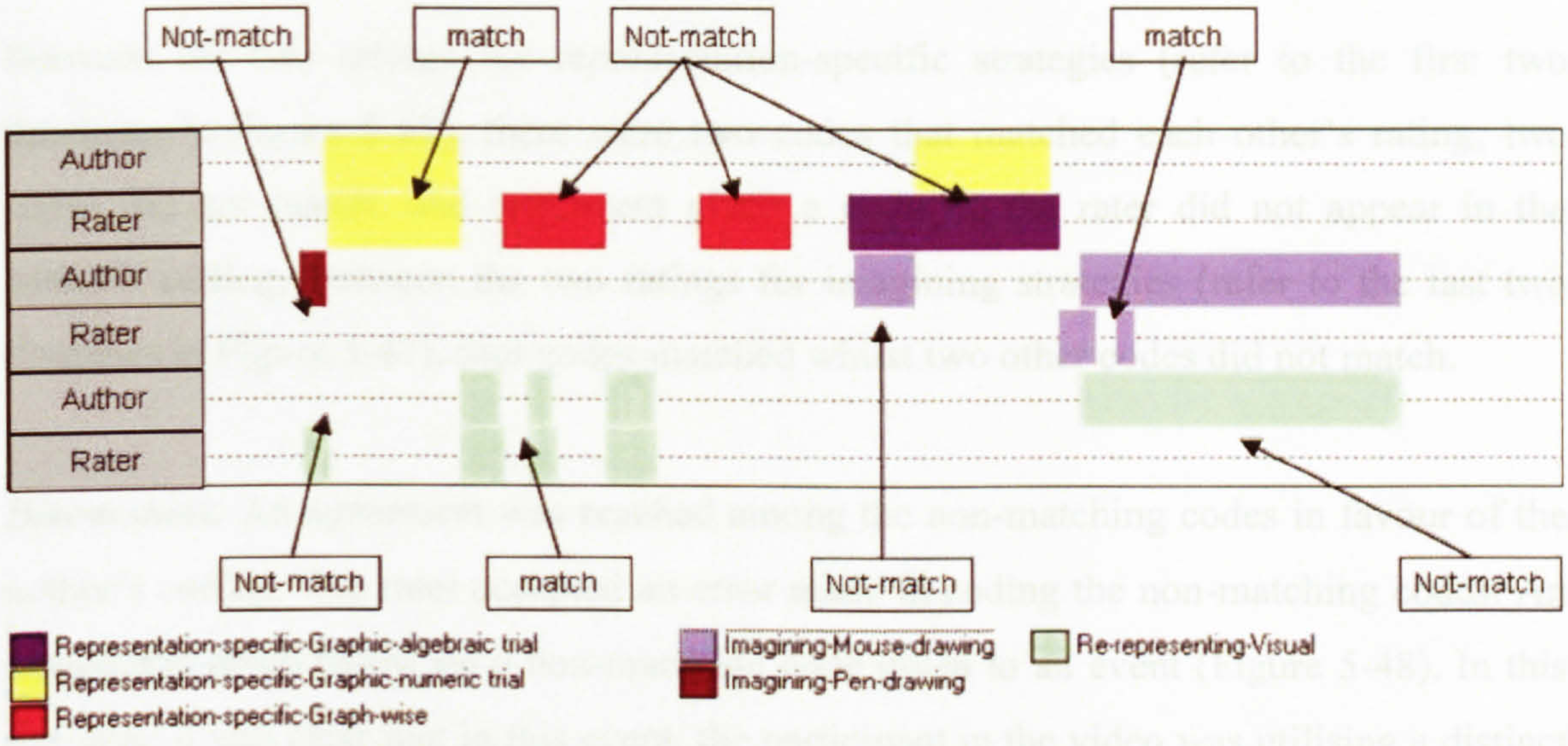


Figure 5-46 Comparison of codings between the author and the rater of the first video

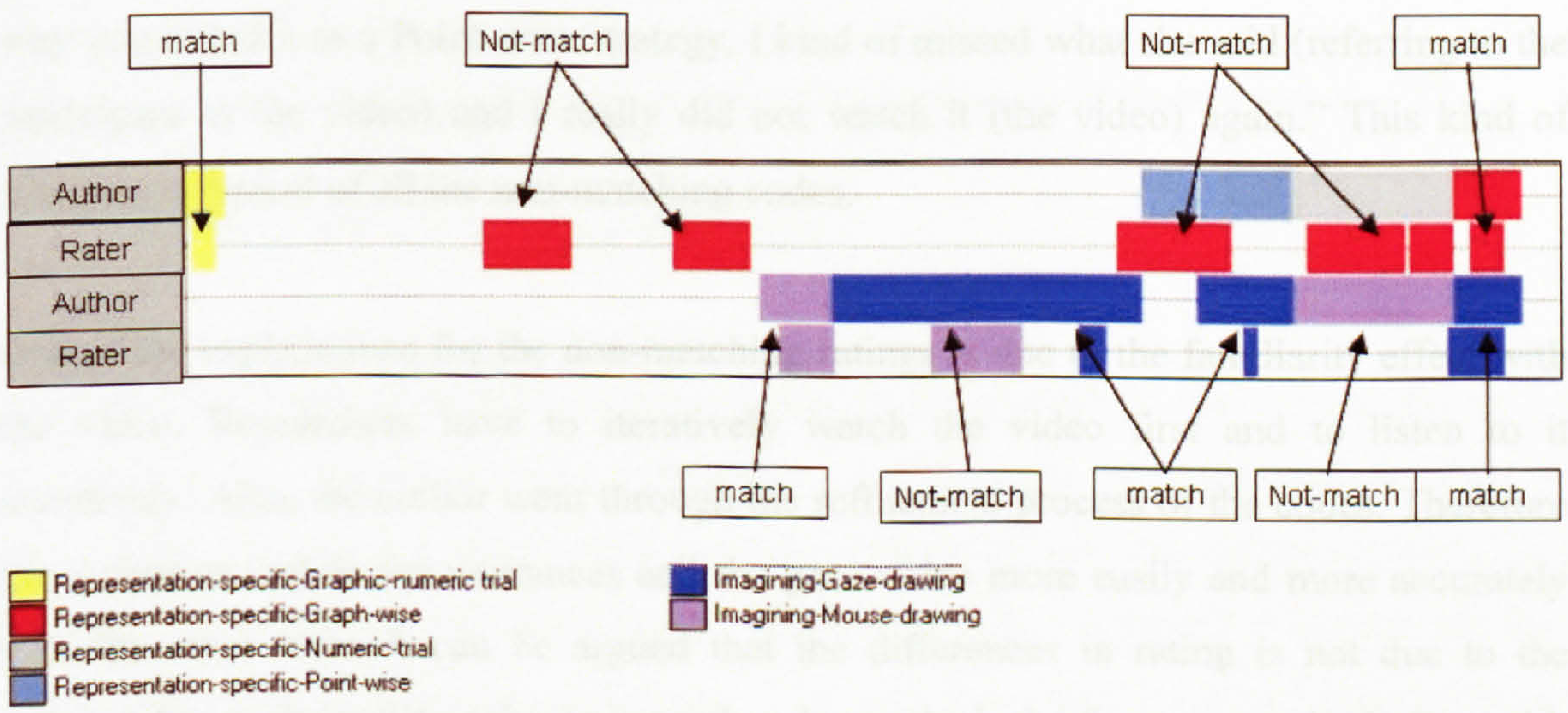


Figure 5-47 Comparison of codings between the author and the rater of the second video

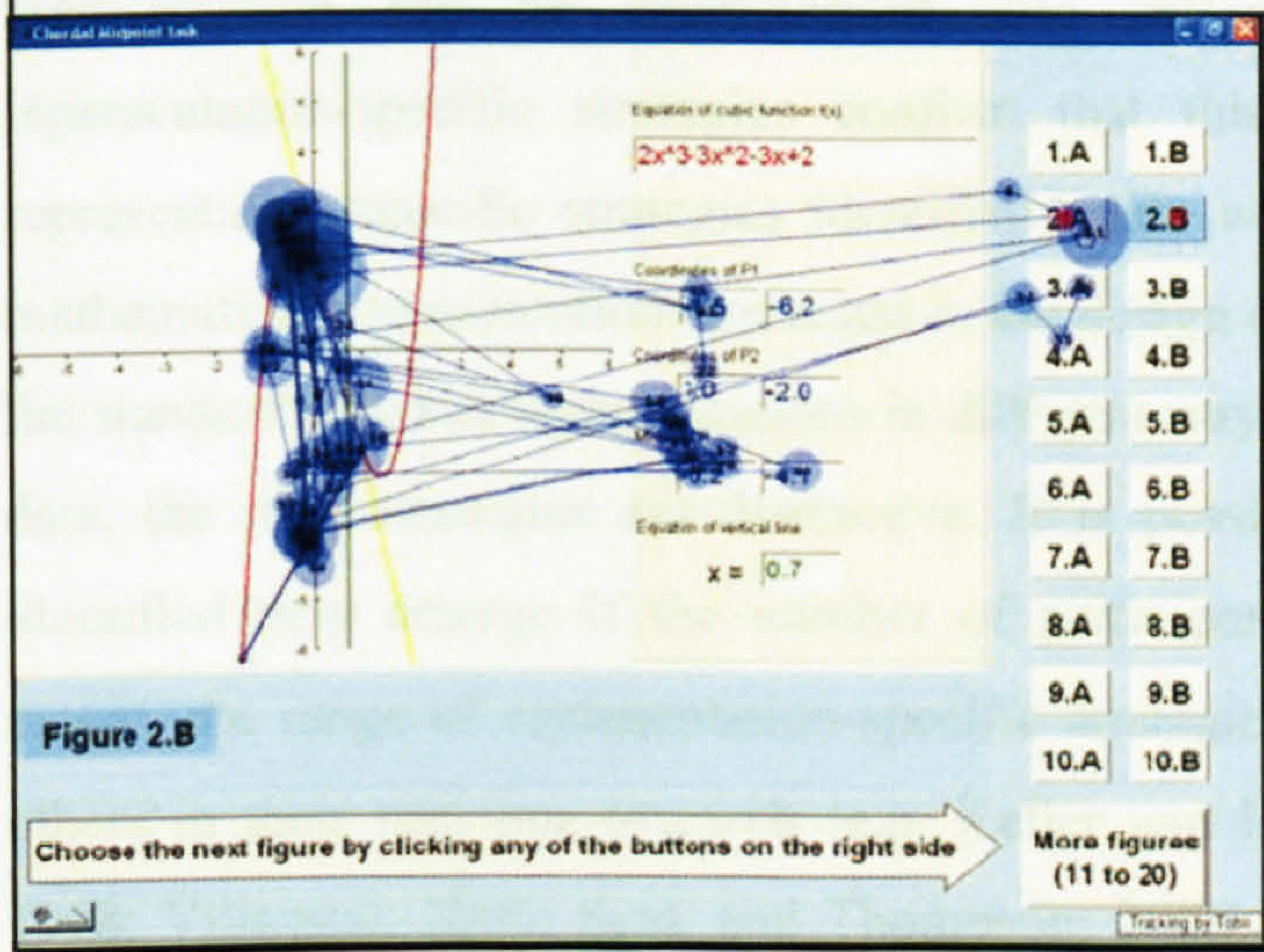
Second video. For each code the author and the other rater disagreed on, they watched the event together (i.e. the part of the video coded). Figure 5-47 shows four Timelines, the first two timelines are the representation-specific strategies coded by the author and the rater, respectively; the other two timelines are for the imagining strategies. Both raters agreed that there were no re-representing strategies used within the video coded.

Between the two ratings for representation-specific strategies (refer to the first two timelines in Figure 5-47), there were two codes that matched each other's rating; two codes did not match; and one event given a code by the rater did not appear in the author's coding. Between the two ratings for imagining strategies (refer to the last two timelines in Figure 5-47), four codes matched whilst two other codes did not match.

Discussions. An agreement was reached among the non-matching codes in favour of the author's coding. The rater accepted an error made in coding the non-matching codes. An example is given below on a non-matching code given to an event (Figure 5-48). In this example, it was clear that in this event, the participant in the video was utilising a distinct point of the graph that is why the author coded this as a Point-wise. The participant in the video being coded did not really describe a graphic behaviour, thus, this event is not a graph-wise strategy. The researcher agreed with the author's coding. She said "I can see why you coded it as a Point-wise strategy, I kind of missed what she said (referring to the participant in the video) and I really did not watch it (the video) again." This kind of situation is typical of all the non-matching codes.

One of the explanations for the non-matching ratings is due to the familiarity effect with the video. Researchers have to iteratively watch the video first and to listen to it attentively. Also, the author went through the refinement process of the codes. Therefore the author can relate the utterances and the gaze video more easily and more accurately than the other rater. It can be argued that the differences in rating is not due to the unreliability of the coding scheme but rather due to the lack of experience in dealing with the video being coded. In the preliminary session mentioned above, the raters may need to be trained by asking them to watch some videos. For example, with the kinds of videos in this research, probably watching three 15-minute videos could be enough to increase accuracy of coding.

Author's code: Point-wise strategy
Researcher's code: Graph-wise strategy



Utterance/s:

"I'm checking where the midpoints are compared to the turning points..." (00:08:44:13)

Description of action: P4 started to talk, gazed were found on the maxima and the midpoint.

Figure 5-48 An example of non-matching codings of the two raters

5.6 SUMMARY AND DISCUSSION

The new approach illustrated in this chapter is an attempt to show how the set of techniques of combining gazes, utterances, actions, and writings can be used to investigate learners' interaction with multiple representations. The techniques provide a more accurate way of analysing learners' strategies with multiple representations than using previous data collection techniques. This was done with the aid of state-of-the-art analysis tools for coordinating and analysing utterances, actions, sketches and gazes. The selection of the categories of strategies aims to address what strategies learners use with representations offered to them; and why learners change their strategies in completing a task.

Five different categories of strategies have been considered namely representation – specific, imagining, re-representing, generic, and interface-supported. The classification of *representation-specific strategies, imagining strategies and re-representing strategies* was a convenient way of categorising the strategies identified to address the hypotheses.

The first hypothesis in this research is that strategies with each standard external representation can be characterised into different levels of granularities. The representation-specific strategies confirm that this is the case. The nine codes of representation-specific strategies identified relate with three kinds of standard external mathematical representations included in the design of the interface. Learners use each of the standard external representations in different ways. As far as can be detected from the data, the nine strategies are distinctive. It is possible that more codes than the ones identified may emerge if the number of participants is increased. However, to some extent, the range of representation-specific strategies match the strategies identified by others in their previous research (e.g. Keller and Hirsch, 1998; Ruthven, 1990; Even, 1998; Villarreal, 2000; Senk and Thompson; 2006, Tabachneck et al., 1994). Also, the characterisation of these strategies suggests that there is a relationship between strategies with standard external representations and difficulties with them. This corroborates Cox's (1996) suggestion that strategies with external representations can provide indications of difficulties.

These innovative ways of analysing and capturing digital videos makes it possible to identify other ranges of strategies in finer granularities than can be identified by traditional observation. Strategies related to imagery and expression of inferences by others (e.g. DeWindt-King and Goldin, 2003; Yoon and Narayanan, 2004; Laeng and Teodorescu, 2002; Goldin-Meadow et al., 2001; Cox, 1996) contribute to the range of imagery and re-representing strategies identified in this research.

This chapter provides evidence for significantly comparing representation-specific, imagining and re-representing strategies. It provides explanation for the hypothesis that strategies with standard external representations can be characterised into different levels of granularities. The next two chapters present the results of comparing these three sets of strategies between tasks and instantiations. Chapters 6 and 7 present quantitative and qualitative analyses of gazes, utterances, actions and sketches. The next chapter examines learners' use of strategies in terms of time to task completion, frequency of occurrences of each strategy, and average of frequency and duration of fixations of attention on AOIs.

6 QUANTITATIVE DATA: A PRELIMINARY ANALYSIS

6.1 INTRODUCTION

After presenting the innovative technique of capturing and the approaches to analysing learners' strategies with multiple computer-based representations, this chapter first applies the approach to analysis on quantitative data. This chapter examines the number of participants using certain strategies, the time spent to task completion and the time spent in looking at specific Areas Of Interest. This chapter presents initial quantitative analysis before delving into the qualitative analysis. By doing this quantitative analysis, this chapter gives some details on identified extraneous strategies that can be excluded in the subsequent analysis.

Many researchers have pointed out the 'task effect' on studies about multiple representations (e.g. Cox, 1996; Leinhardt et al., 1990; Kozma, 2003; Ainsworth, 2006). The aims of this form of analysis are: to compare the time spent in completing tasks and see if an association exists in varying instantiations – Static, Dynamic and Interactive; to compare the frequency of participants who answer the task correctly and relate this to time spent with the task between instantiations; to compare the amount of fixation on each type of representation for each task between instantiations; to come up with a list of strategies that the participants of the study used; and to identify extraneous strategies that can be excluded in the analysis that is discussed in chapter 7. The findings are then discussed in the light of other claims relating to the effect on time to task completion and performance in varying instantiations.

6.2 TASK COMPLETION

The summary of findings related to time to task completion and number of participants with correct answers is presented first. Some possible explanations about the findings, the time to task completion and frequency of correct answers are given, and are related to some of the studies identified in the literature.

Main findings

There are three main results relating to task completion. First, there appears to be no overall relation between the type of instantiation and the average time spent on a task, although for any given task there are differences in time spent on different instantiations. For example, in one particular task, participants in Static spent less time than participants in non-Static, however this was not true for the other two tasks. In another task, the participants in Interactive spent less time than participants in non-Interactive. It is found that one task generally takes less time than the other two tasks; and almost all of the participants managed to answer this task correctly. Second, for each task, whilst the average time spent varies between instantiations, the total number of participants with correct answers does not vary much between instantiations. For example in one of the tasks, the average time it takes participants to finish the task in Static is about 10 minutes whilst in non-Static it takes about 15 minutes. However, the number of participants with correct answers on that task is almost equal in each instantiation. Third, the tasks are not equally difficult.

Time to completion of tasks

The time that participants spent in the tasks are presented in Appendix B. Table 6-1 shows the averages of time spent in completing each task between instantiations. The overall average time varies between tasks (Root task – 00:13:34:02, Chord task – 00:17:54:11; Tangent task – 00:17:24:02). The Root task has the least average time spent in completing the task than the other two tasks. The table also shows that the average time spent within each task varies between instantiations. In the Root task, the participants spent longer time in Static than in non-Static; in the Chord task, the participants took longer time in Static than the non-Static; and in the Tangent task, it seems that the participants took lesser time in Interactive than in non-Interactive. The average of time taken in the Root task is less than the time taken with the other two tasks. This suggests that the analysis between tasks should be considered in terms of the levels of difficulty among them. The time and the correct responses (discussed below) between tasks suggest that the Root task can be referred to as a 'simple' task whiles the other two tasks are 'complex'.

Table 6-1 Averages of time spent to task completion

	Root task	Chord task	Tangent task
Static	00:09:22:22	00:20:04:18	00:17:10:06
Dynamic	00:16:36:14	00:16:03:12	00:21:50:12
Interactive	00:14:52:22	00:17:35:05	00:13:11:14
Overall Average	00:13:34:02	00:17:54:11	00:17:24:02

Frequency of correct answers

$y = 2a - x$

where x = roots of $f(x)$

y = roots of $g(x)$

a = point rotated about.

Figure 6-1 An example of a 'correct' generalised conjecture

The expected generalised conjectures for each of the task are given in **Table 4-2** (the tasks). The participants' answers are considered 'correct' when the answers are similar to the meaning of the expected generalised conjectures. The answers do not necessarily come exactly the same as the expected generalised conjecture. For example in the Root task the expected answer is "the new roots are reflections of the roots of the original. The same distance from point (a, 0)". An answer, similar to the one in Figure 6-1, is in algebraic notation. This answer is equivalent to the expected generalised conjecture. All the answers marked as correct are given in Appendix E.

Table 6-2 shows the number of participants with correct answers in each task between instantiations. It shows that correct answers vary between tasks. 15 out of 18 answered the Root task correctly, 0 out of 18 in the Chord task, and 7 out of 18 in the Tangent task. Also, in the Tangent task, more participants correctly answered the task in non-Static than in Static. This result seems to suggest that the Root task appear to be the easiest among the three tasks. It also seems that the Tangent task tends to be easier when the instantiation is non-Static.

Table 6-2 Frequency of participants who answered the task correctly

	Root task	Chord task	Tangent task
Static	5/6	0/6	1/6
Dynamic	4/6	0/6	3/6
Interactive	6/6	0/6	3/6
	15/18 (83%)	0/18 (6%)	7/18 (39%)

6.3 FREQUENCY OF PARTICIPANTS FOR EACH STRATEGY

The aim of the analysis in this section is to identify strategies associated with each task. It is valuable to understand the kinds of strategies that participants use in each task as suggested by Cox (1996, 1999). The data were coded for task, instantiation, and for each category of strategy (i.e. representation-specific, imagining, and re-representing strategy), in order to identify associations between strategy, instantiation and task. The numbers of participants using certain strategies are arranged in a 'panelled' column chart as described earlier in chapter 5 section 4. By inspecting the panelled chart, it is possible to distinguish which strategies appear within each task. It also helps differentiate the strategies appearing across instantiations. The main finding is presented first followed by the evidence in a panelled chart for each category of strategy. All of the strategies chosen by the participants are shown in Appendix C.

Main finding

The participants' choices of strategies vary between simple and complex tasks. For example, in the two complex tasks (Chord and Tangent), when participants use only graphic-related strategies there is no evidence of symbolic re-representing strategies; but there is evidence of imagining strategies. Meanwhile in the simple task, the Root task, when participants use algebraic-related strategies there is evidence of symbolic re-representing but less evidence of imagining strategies.

Representation-specific strategies

Figure 6-2 shows that the Root task, the simple task, elicited all of the representation-specific strategies. The 'arrows', in Figure 6-2, points to the strategies being compared between instantiations. The participants in Dynamic Root task used more algebraic-related strategies than participants in non-Dynamic instantiations. More participants used algebraic-related strategies in Dynamic than in Static or Interactive. Further analysis on the occurrences of algebraic-related strategies is presented in the next chapter. The column chart also shows that graphic-related strategies are dominant in the Chord and the Tangent tasks. One graphic-related strategy, the point-wise strategy, was not apparent in Dynamic-Chord task. Many participants used the point-wise strategy in Static and in Interactive. The small number of participants does not provide sufficient evidence to claim that a particular instantiation encourages certain strategies (specific to a particular external representation) no matter what the task is. However, the numbers of participants using a certain strategy vary between instantiation when the task is the same.

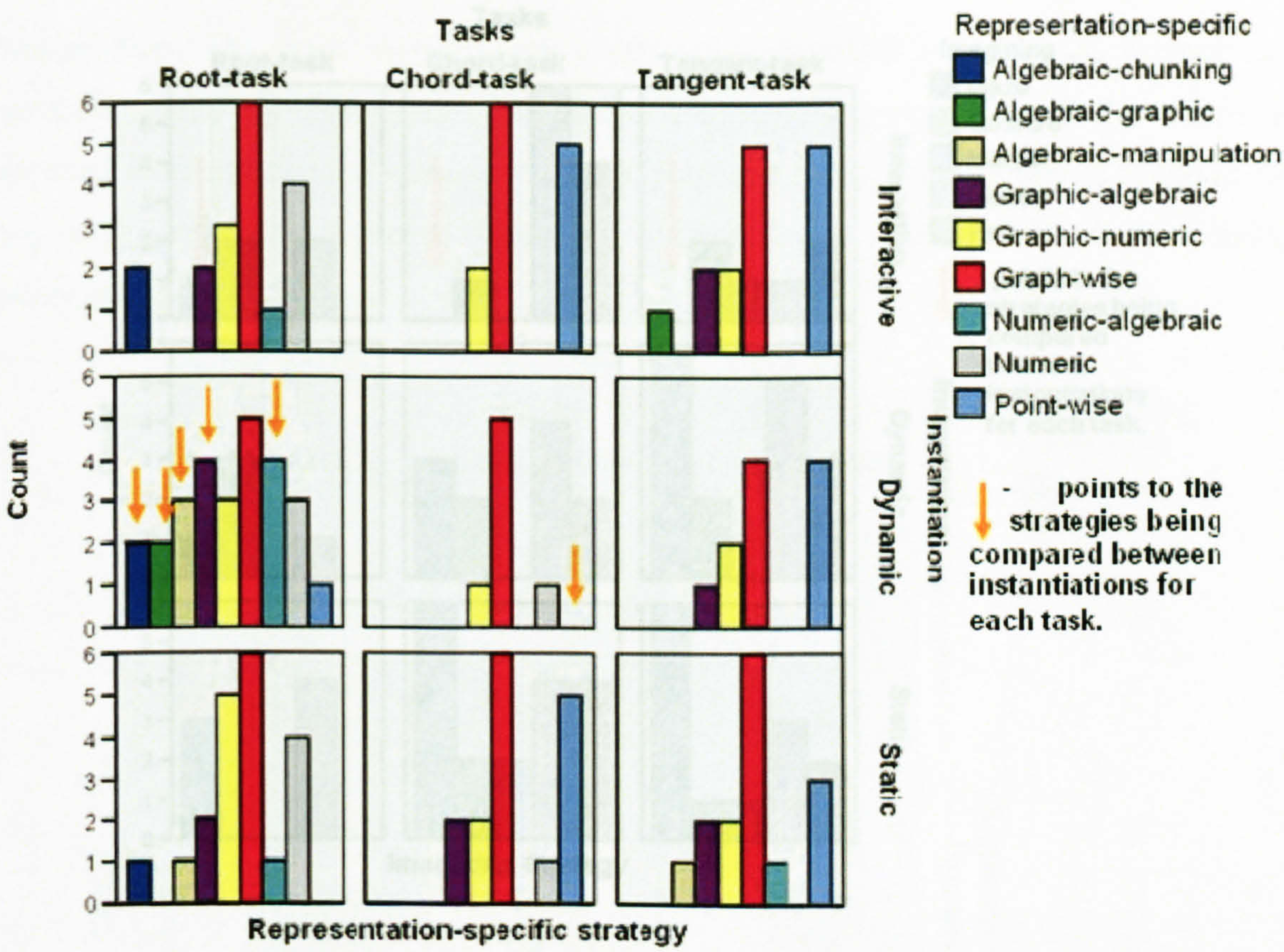


Figure 6-2 Frequency of participants using certain representation-specific strategies

Imagining strategies

The combination of eye-tracking, tablet PC screen capture, think-aloud, and video recording made it possible to identify imagining strategies that related to how learners externalised mental images, because it allowed the analyst to relate actions and utterances accurately to precise gaze in any given moment. In this section, participants' imagining strategies are compared similarly to the way representation-specific strategies are analysed above. The aim is to find out whether some imagining strategies are associated only with certain types of instantiation regardless of the task and to verify whether participants use different imagining strategies between instantiations of the same task.

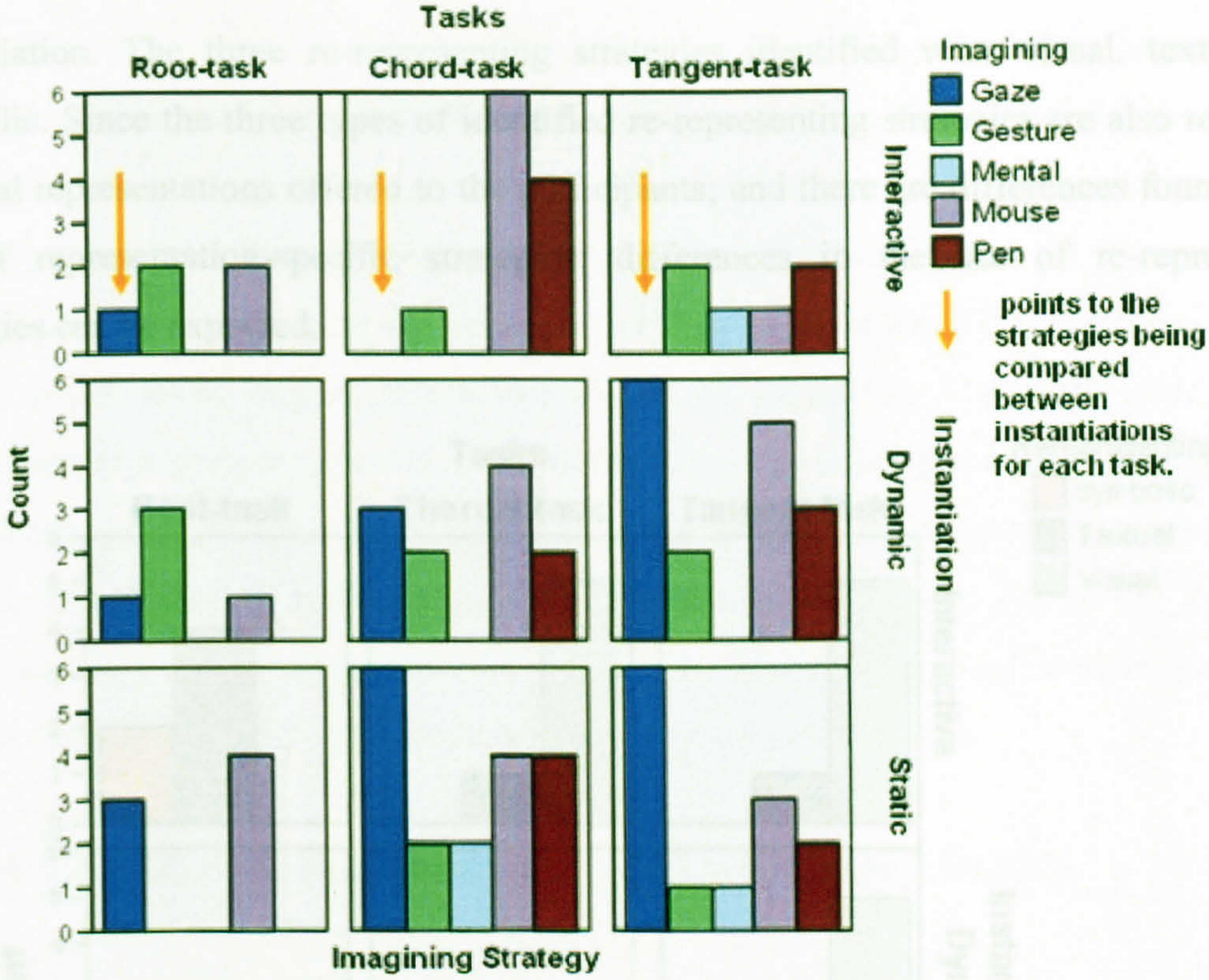


Figure 6-3 Frequency of participants using certain imagining strategies

Figure 6-3 shows that varying the instantiation *does* seem to make a difference in imagining strategies. Imagining strategies seem to be less common in Interactive instantiation. In particular, gaze drawing did not appear to be used at all in Interactive instantiation. It is obvious from the column chart that there are more participants who used gaze-drawing strategy in the Static and the Dynamic instantiation than in Interactive. Gaze imagining strategy was decreasing from Static (15 out of 18), to Dynamic (10 out of 18), to Interactive (1 out of 18). Possible explanations as to why gaze-drawing strategies were less in Interactive are presented in the next chapter.

Re-representing strategies

The third category of strategy analysed here is learners' use of re-representing strategies. The analysis above shows that there are differences in the number of participants using some representation-specific strategies and imagining strategies by task and by instantiation. The three re-representing strategies identified were visual, textual and symbolic. Since the three types of identified re-representing strategies are also related to external representations offered to the participants; and there are differences found in the use of representation-specific strategies; differences in the use of re-representing strategies can be expected.

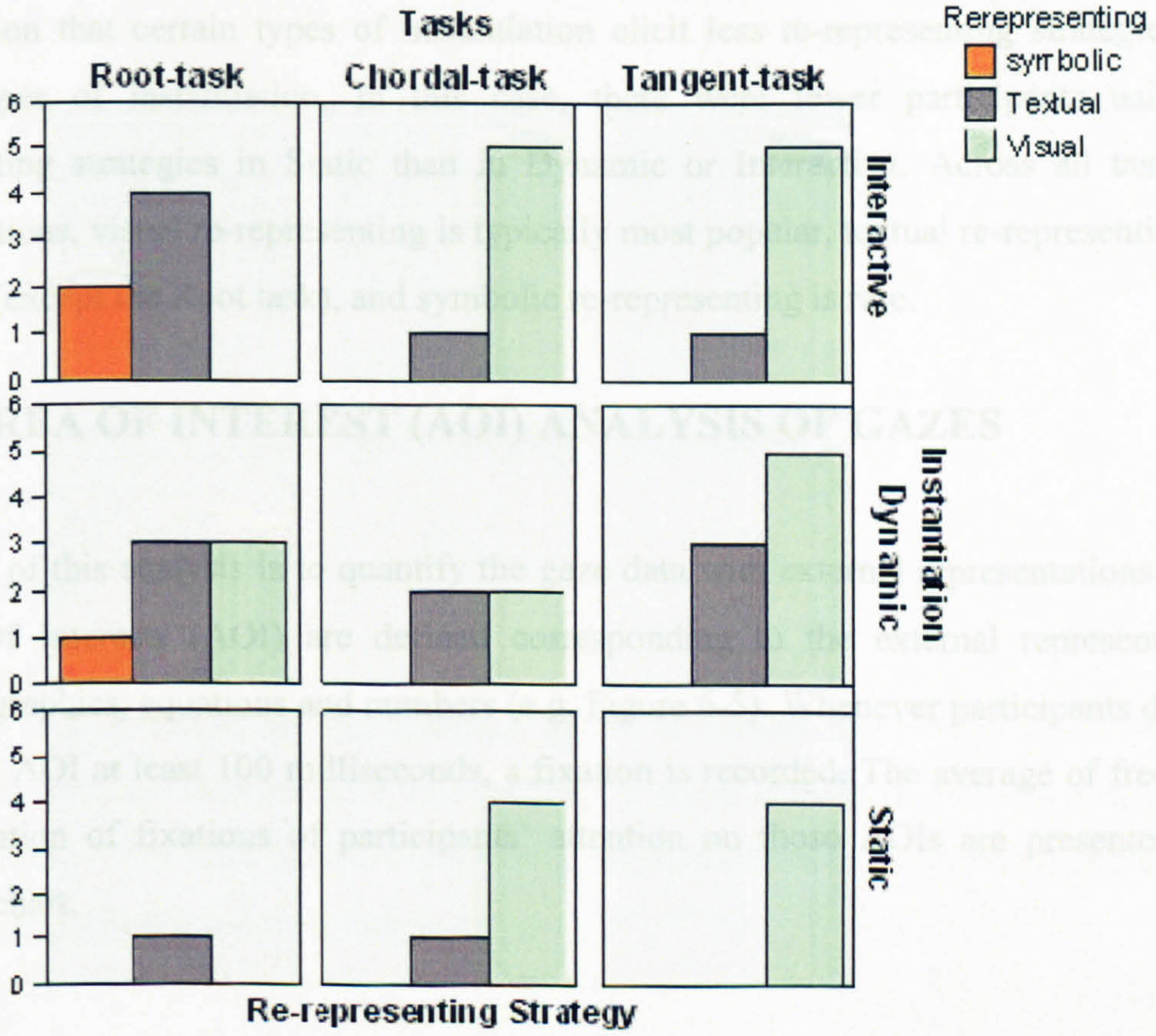


Figure 6-4 Frequency of participants using certain re-representing strategies

Figure 6-4 shows the number of participants who used re-representing strategies by task. The symbolic-re-representing strategy was only found in the Root task. The visual re-representing is more prominent in the Chord and Tangent tasks than in the Root task. There were more participants using the visual-re-representing strategy in the Chord task (11 out of 18, 61%) and the Tangent task (14 out of 18, 78%) than the Root task (3 out of 18, 17%). The analysis of the number of participants who used re-representing strategies by instantiation also shows that there is a marginal difference in the frequency of participants who used visual re-representing strategies in the three instantiations. Although no one used a symbolic re-representing strategy in Static the difference is very small and clearly insignificant. Also, it was already identified above (Figure 6-4) that the symbolic appeared only under the Root task. On the other hand, only 11% (2 out of 18) of the participants used textual re-representing strategy. This is small compared to Dynamic (8 out of 18, 44%) and in Interactive (6 out of 18, 33%). The result confirms the expectation that certain types of instantiation elicit less re-representing strategies than other types of instantiation. In this case, there were fewer participants using re-representing strategies in Static than in Dynamic or Interactive. Across all tasks and instantiations, visual re-representing is typically most popular, textual re-representing less popular (except the Root task), and symbolic re-representing is rare.

6.4 AREA OF INTEREST (AOI) ANALYSIS OF GAZES

The aim of this analysis is to quantify the gaze data with external representations. Three Areas Of Interests (AOI) are defined corresponding to the external representations, namely graphics, equations and numbers (e.g. Figure 6-5). Whenever participants dwell at a certain AOI at least 100 milliseconds, a fixation is recorded. The average of frequency and duration of fixations of participants' attention on those AOIs are presented on a column chart.

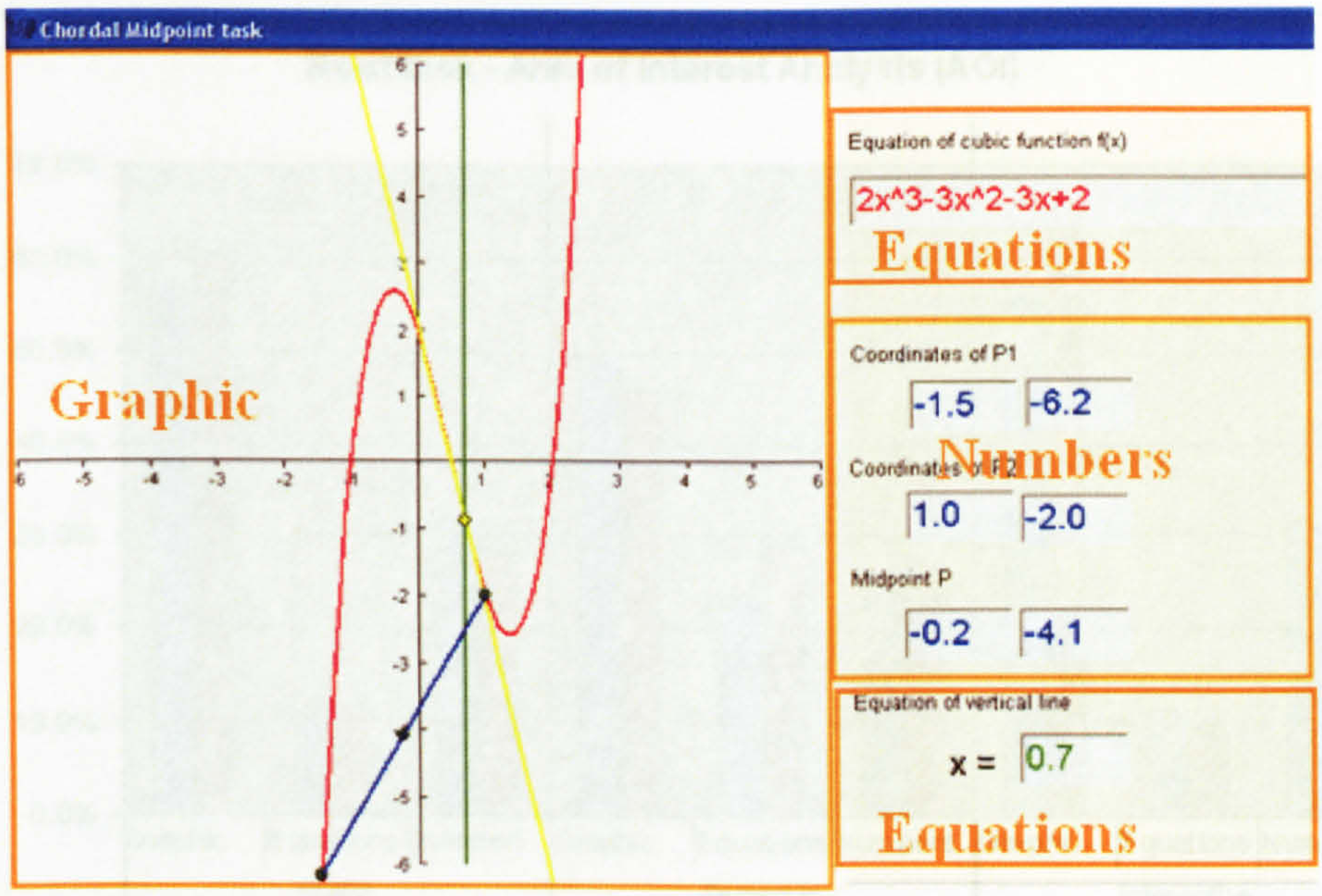


Figure 6-5 Areas Of Interest

Main finding

An instantiation can direct participants' gazes to certain representations. There is a difference between instantiations in how long is spent looking at each type of external representation. The AOI analysis shows that attention to external representations varies between instantiations within each task.

Root task

Figure 6-6 shows the column chart of the AOI analysis for the Root task. The participants focused mostly on the graphic representations across instantiations. The participants spent at least 50% of their time looking at the graphs in all three instantiations. Participants in non-Dynamic seemed to pay more attention to the numbers than the equations. The frequency and durations of fixations showed at least 10% were on equations whilst at least 30% were on numbers. On the other hand, participants paid attention to numbers and equations in Dynamic within the range of 23% to 29%.

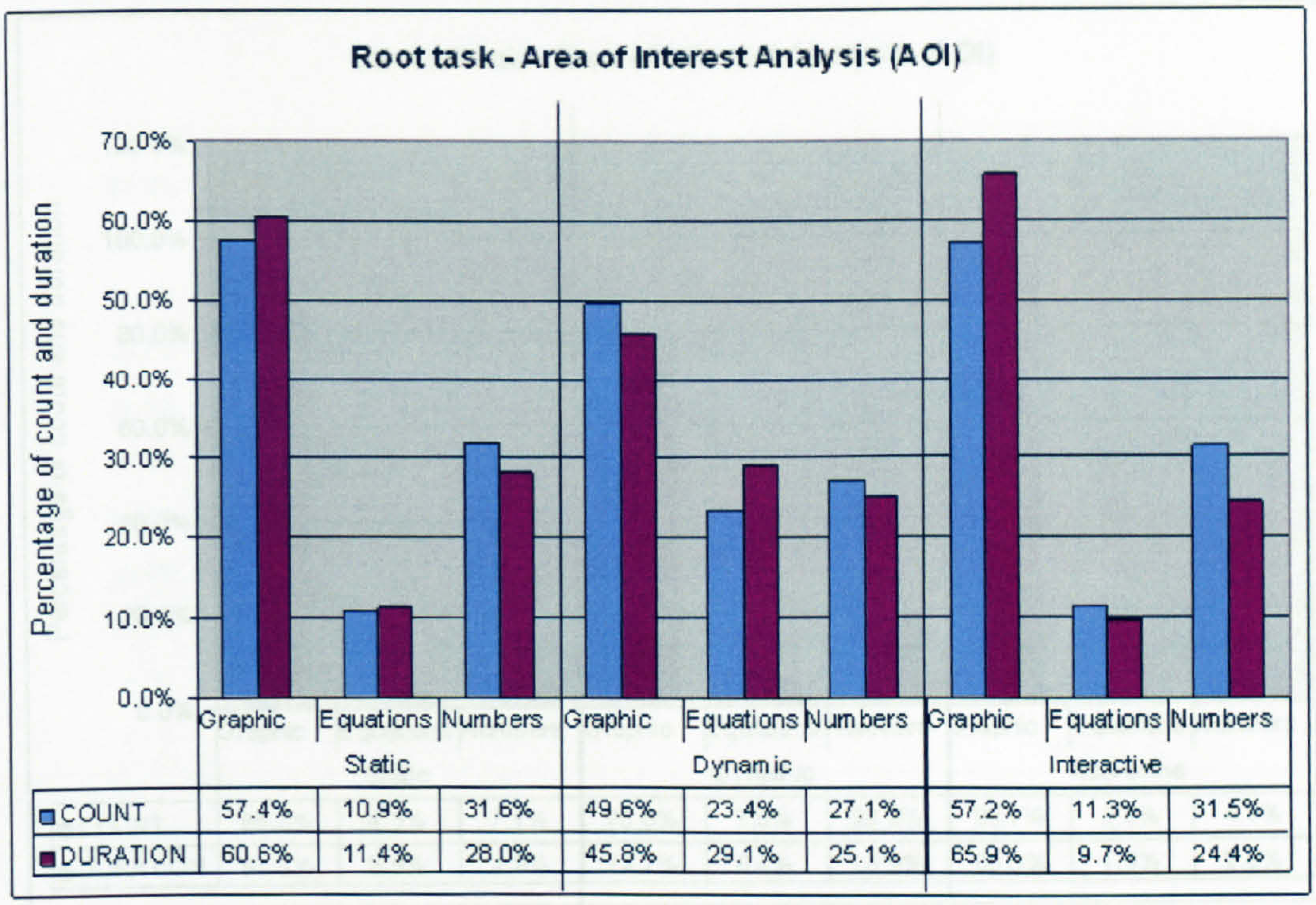


Figure 6-6 AOI analysis for the Root task

Chord task

For the Chord task, the AOI analysis shows participants' fixations seemed to concentrate on the graphic representations in all three instantiations (Figure 6-7). More than 70% of the fixations were recorded on graphic representations. The fixations on numbers and equations seemed to vary between Dynamic and non-Dynamic instantiations. In non-Dynamic, the difference among the fixations on equations and numbers was marginal, ranging from 3.5% to 7% in Static, whereas Interactive ranged from 1.9% to 4.1%. In Dynamic, the difference in fixation was about 7% on equations whilst at least approximately 19% on numbers.

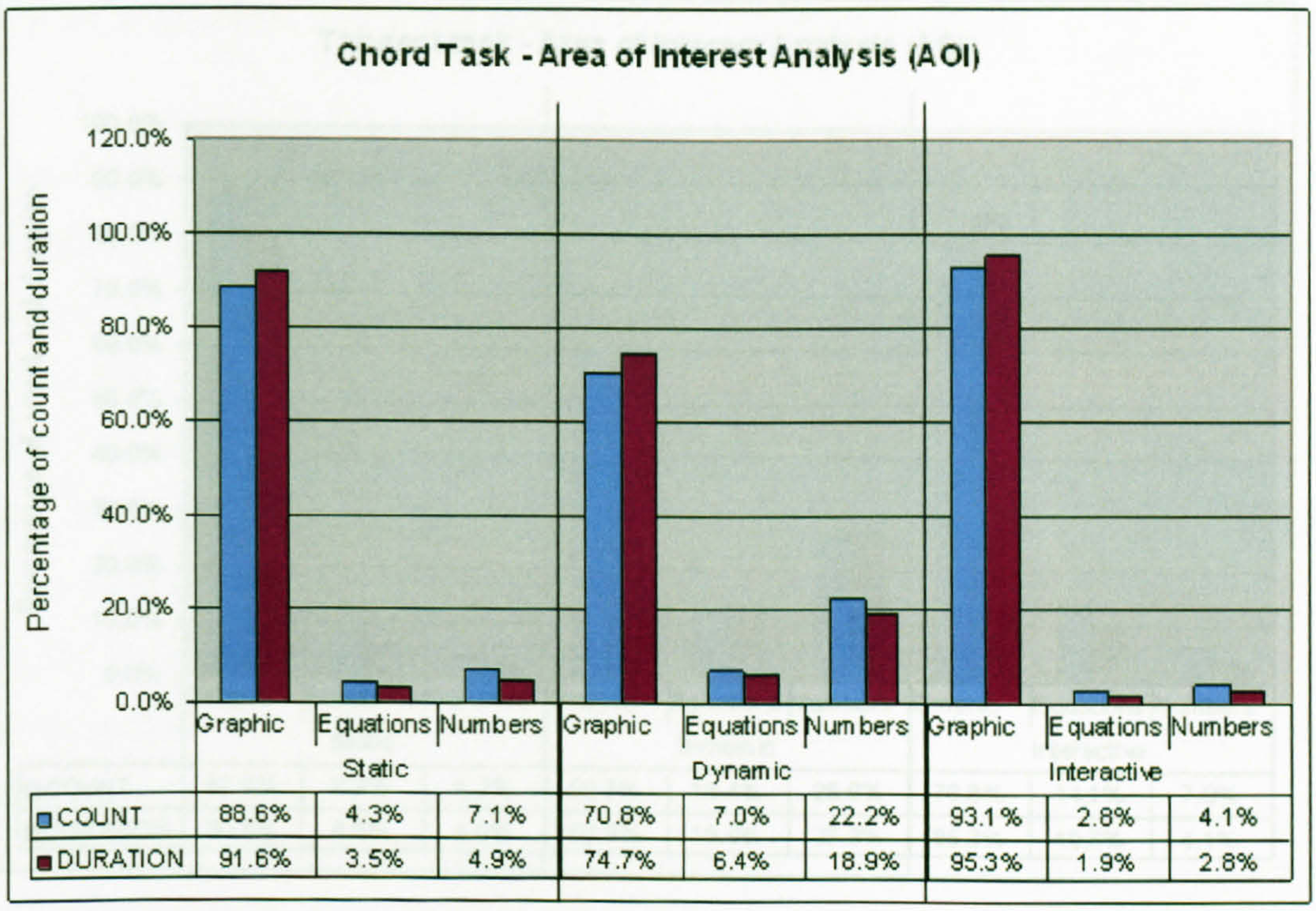


Figure 6-7 AOI analysis for the Chord task

Tangent task

The frequency and durations of fixations for the Tangent task (Figure 6-8) seemed to a show similar results to Chord task. The attention mostly focused on the graphic and a marginal difference was found in frequency and durations of fixations on equations and numbers in non-Dynamic whilst a variation was found in Dynamic.

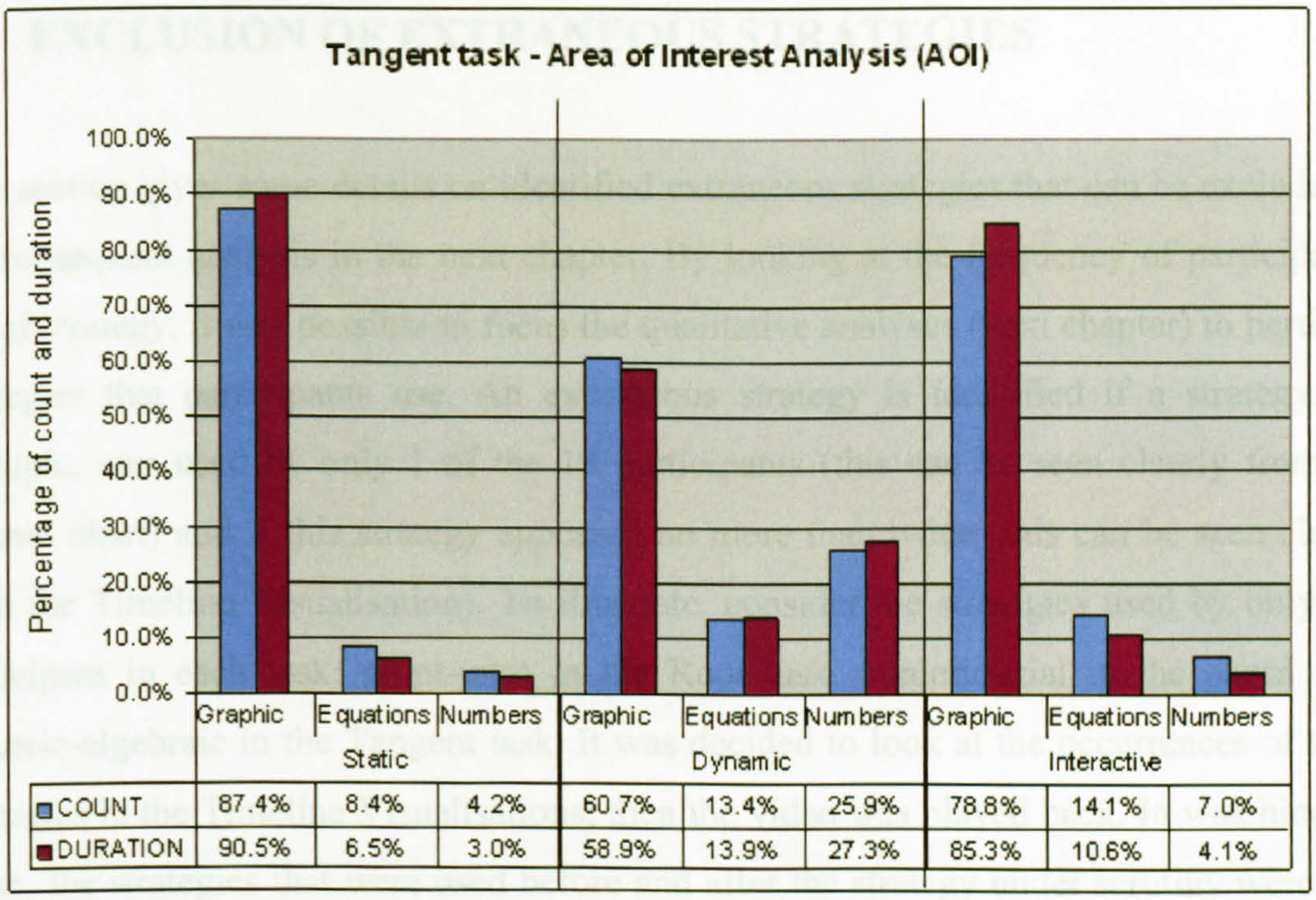


Figure 6-8 AOI analysis for the Tangent task

Hotspot Visualisation

The Hotspot Visualization summarises participants' attention. Under the Root task, the Hotspot Visualisation (Figure 6-9) shows that in Static and Interactive, participants did not seem to pay attention to one of the equations. For the Chord and the Tangent tasks, the Hotspot seemed to appear darker in the number areas than in the equation areas. Across instantiations, the Hotspot mark on the graphical representation appeared to cover the same area.

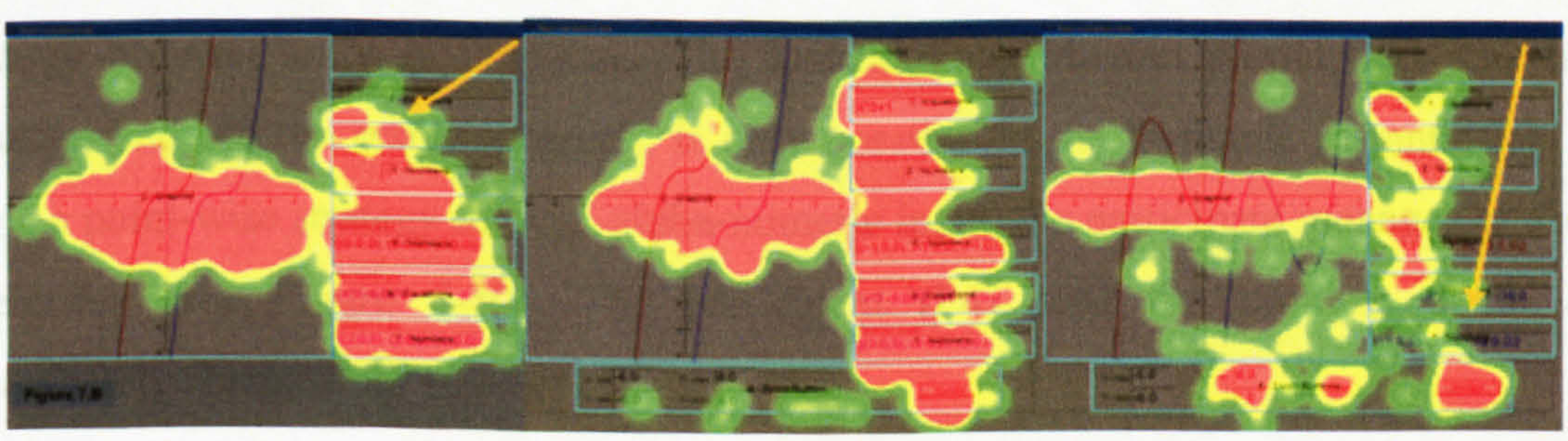
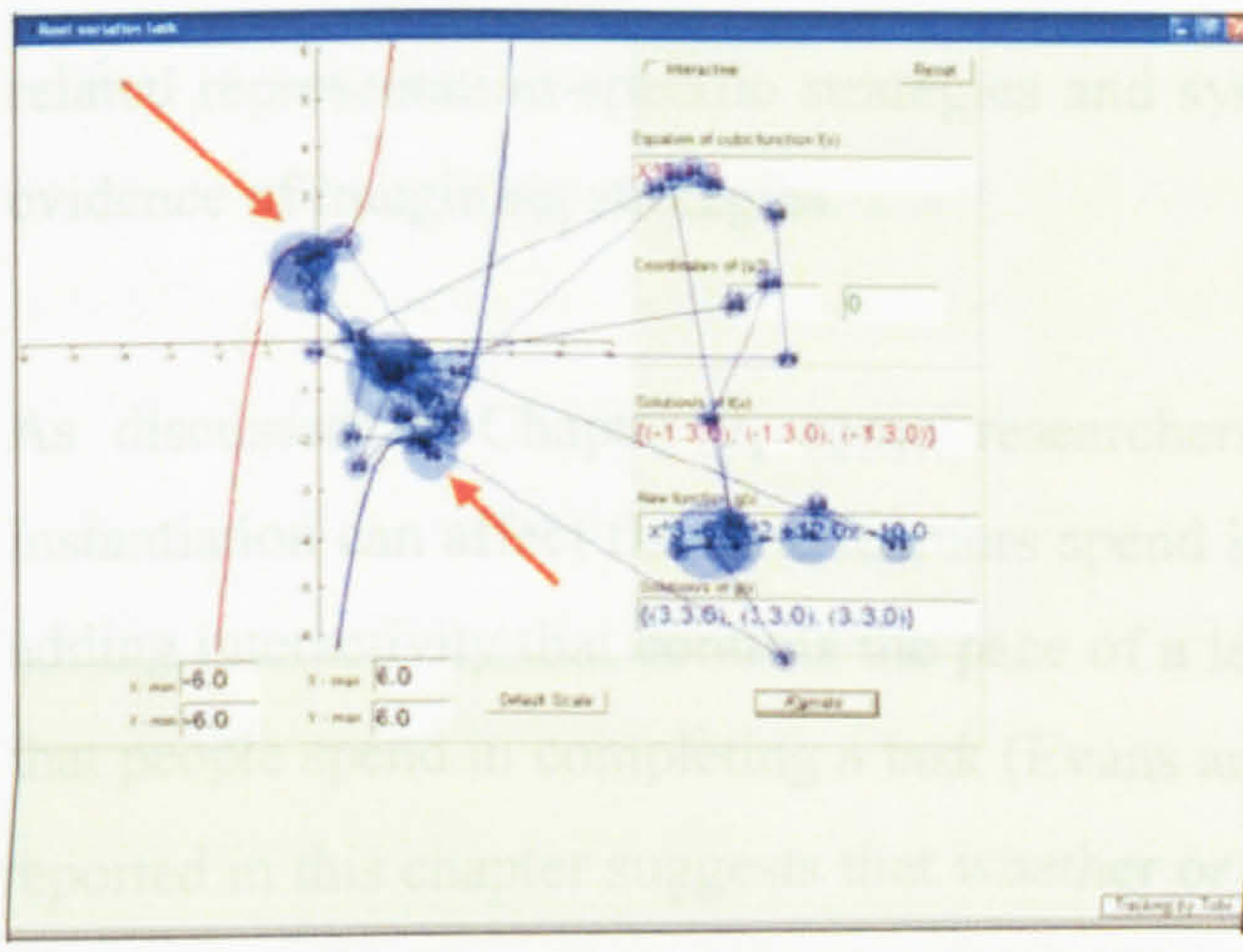


Figure 6-9 Hotspot Visualisation for the Root task

6.5 EXCLUSION OF EXTRANEOUS STRATEGIES

This section gives some details on identified extraneous strategies that can be excluded in the subsequent analysis in the next chapter. By looking at the frequency of participants' use of strategy, it was possible to focus the qualitative analyses (next chapter) to pertinent strategies that participants use. An extraneous strategy is identified if a strategy, for example, was used by only 1 of the 18 participants (this can be seen clearly from the column chart) and if this strategy appeared no more than twice (this can be seen clearly from the Timeline Visualisation). To illustrate, consider the strategies used by only one participant in each task: point-wise in the Root task, numeric-trial in the chord task, numeric-algebraic in the Tangent task. It was decided to look at the occurrences of these strategies in the Timeline Visualisations; then the video was played back. In watching the video, the strategies that were used before and after the strategy under scrutiny were also looked at in order to see whether there is a connection between the previous or the subsequent strategy. This is the basis for deciding whether the strategy was likely being seen by the participant as not useful in the task being tackled.

An example is given in Figure 6-10. A point-wise strategy was used by only one participant and occurred in the Timeline once. It was therefore presumed that this strategy was considered by the participant as irrelevant. In Figure 6-10, the point used in this instance is the point on the y-axis. This strategy used a representation that is not helpful to answer the task. The representations that may help are: to compare either the graphical points found on the x-axis, or to use the numerical coordinates or to use the two equations and the point of rotation. By looking at the strategies before and after this strategy, it was possible to tell that this strategy was dismissed at once. This particular extraneous strategy occurred only in Dynamic. The instantiation, wherein participants can enter equation and see the resulting graph, is only possible in the Dynamic instantiation. By using a point-wise strategy through alpha-numeric input, the participant manages to check inferences (see Figure 6-10). The participant (P4) read the problem again and immediately changed her strategy.



Dynamic

00:05:25:01

(P4 related the rotation in terms of the turning point which is a discrete point of a cubic function)

P4: It is still two zero but it got shifted two points that way

Figure 6-10 An excluded strategy in the analysis

There were only a few extraneous strategies identified by this analysis: the point-wise strategy illustrated above, and algebraic-graphic and numeric-algebraic strategies in the Tangent task. However, these strategies are not excluded in analysing changes in strategies chosen by each and every participant. This strategy may give indications of difficulty in the course of answering the tasks.

6.6 INTERPRETATION AND DISCUSSION

This chapter examined time spent on tasks, successful completion of tasks by counting the number of participants who gave correct answers, and the number of participants' choice of each strategy on three tasks between three types of instantiations. The analyses suggest that the three tasks have different levels of difficulties. The Root task is a 'simple' task because more participants were successful in this task than in the other two tasks, and participants took less time on this task than on the other two tasks. Thus, the other two tasks, Chord and Tangent tasks are complex because participants took a long time and only a few of the participants successfully completed the tasks. The participants' choices of strategies varied between simple and complex tasks. In complex tasks, participants used graphic-related representation-specific strategies and imagining strategies but did not use a symbolic re-representing strategy. In the simple task, participants used algebraic-related representation-specific strategies and symbolic re-representing but there was less evidence of imagining strategies.

As discussed in Chapter 2, some researchers have suggested that certain types of instantiation can affect the time learners spend in completing a certain task. In particular, adding interactivity that controls the pace of a learning task is reported to reduce the time that people spend in completing a task (Evans and Gibbons, 2006). However, the research reported in this chapter suggests that whether or not users can move representations about the screen, and whether or not the representations are manipulable are not good predictors of time to task completion. The analyses of time for each task over three instantiations shows mixed results. For a simple task, when the task is in Dynamic and Interactive, participants spent more time than in Static. For one of the complex tasks (the Chord task), participants took longer when an Interactive feature was present than without. However, in the other complex task (the Tangent task), the longest time participants took was in Dynamic (i.e. animated graphs/graphs moving about the screen), followed by the time taken in Static; and lowest time in Interactive (manipulable graphs). One interpretation of these results is that interactivity can influence the time learners spend in completing a task depending on whether interaction reduces or adds some cognitive load. Increased time to task completion may result when learners interact with an external representation which they have difficulty processing mentally. However, adding interactivity may sometimes reduce the cognitive demand and counterbalance that effect. Further evidence and discussion about this are presented in the next chapter.

This chapter also analysed performance of participants. For each task, the evidence shows that learners' performance did not vary much between instantiations. The findings are in line with other findings of Evans and Gibbons (2006) that learners' performance in a task is not related to the time that learners spend in doing the task. In the three tasks used in this study, the number of participants with correct answers varies between tasks but does not vary much for each task with different instantiations.

Making representations move, manipulable, or requiring alpha-numeric input may lead to learners using a strategy related to the said representation. Learners may have been using a strategy that may be easier to use with a certain instantiation. Thus learners may have arrived at either a correct or wrong answer by applying strategies differently. Further examination is needed in comparing learners' strategies. Comparing the frequency of participants' use of strategies related to external representations and strategies related to imagining and re-representing, provides some insights into this. In the complex tasks, where participants are required to manipulate graphical representations, they also use imagining strategies and re-representing strategies related to graphical representations. The results also suggest that when an instantiation helps them manipulate graphical representations some imagining strategies are not used. For example, the results show that, for a task that requires manipulation of graphical representations, participants use imagining strategies; but when an Interactive feature is present, thus allowing them to manipulate representations instantly, participants do not use such imagining strategies. Interestingly, one imagining strategy, gaze-drawing, was not apparent when the instantiation was Interactive but was found in Dynamic and Static. Learners were found to spend more time in doing a task when they were using visual imagery (c.f. Yoon and Narayanan, 2004). The findings in this thesis seem to corroborate this claim. However, this research suggests that when participants must manipulate the external representation, interactivity affects whether or not they use imagining. When interactivity reduces visual imagining, time to task completion is reduced. However, there is not enough evidence to suggest whether this positively influences successful task completion – or indeed whether it affects learning.

The analyses of gazes suggest that an instantiation can direct participants' gazes to certain representations. There are claims that learners may have difficulty managing attentional resources in a multi-representational environment (e.g. Lowe, 2003; Mayer and Moreno, 1998). Dynamic instantiation in particular can lead to learners paying too much attention to graphical representations and less attention to other representations (c.f. Otero et al., 2001). It is important to clarify whether this difficulty relates to standard external representations or to a particular type of instantiation. The analyses of AOI seem to show a more specific finding than that of Otero et al. This research shows that there is a difference between instantiations in how long is spent looking at each type of external representation. In all three tasks, participants in this research paid more attention to graphic representations than to numeric or algebraic representations. However, comparison of AOI between instantiation shows that participants spent more time looking at graphic representations in Static and Interactive than in Dynamic. The amount of time participants in this study spent looking at each of the different representations (i.e. numerical, graphical and algebraic) were examined. The results show that participants' attention to different representations varies between instantiations. This suggests that instantiation can influence attention. It seems that a particular interactive feature, possibly manipulabilty of graphical representations can direct learners' attention to an external representation. However, AOI analysis also suggest that introducing interactivity such as requiring participants' alpha-numeric input can also direct learners' attention to other non-graphical representations. Chapter 7 offers further evidence about this.

There is much to be said about the multiple strategy effect suggestions of Tabachneck et al. According to these researchers, using multiple strategies is more effective in completing task successfully than using only a few. In certain respects, the findings in this chapter corroborate this. More of the participants of the study answered the Root task correctly than the Chord and Tangent tasks. The analysis of frequency of strategy use suggests that more strategies were used in the Root task than in the other two complex tasks. The tasks are distinguished into simple and complex because of the level of difficulties between the three tasks. Since not many participants completed the Chord and Tangent tasks successfully, these tasks are identified as complex; while the other task as simple. The comparison of representation-specific strategy use and re-representing vary between the simple and complex tasks. This corroborates Cox's (1996) suggestion that re-representing strategies can provide indications of difficulty. The occurrences of re-representing strategies with the two complex tasks are more than those with the simple task. Further qualitative analyses, in the next chapter, can shed light onto this.

In examining occurrences of strategies, this chapter also assists in identifying extraneous strategies. These extraneous strategies have also been qualitatively analysed. In line with Aczel's (1998) suggestion, learning may occur possibly because learners try to improve strategies. The analysis of extraneous strategies in section 6.5 has offered evidence that instantiation plays a role in improving a strategy. This is further examined in the next chapter.

This initial quantitative analysis has helped in identifying the strategies that vary between instantiations when the task is the same. The following chapter reports qualitative analysis undertaken to address the other research hypotheses more fully:

- Learners' choice of strategies depends not just on the standard external representations given but also on the instantiation.

- Mental constructions of images with graphical representations vary between instantiations.

- Attention paid with each standard external representation varies between instantiations.

- Expression of inferences varies depending on the instantiation.

Possible answers to these questions are presented in the next chapter.

7 INTEGRATED ANALYSIS OF GAZES, ACTIONS, UTTERANCES AND SKETCHES

7.1 INTRODUCTION

The previous chapter presented evidence that there were certain differences in time spent to task completion, numbers of participants with correct answers, frequencies with which participants used certain strategies, and durations of time spent looking at each external representation. This chapter presents qualitative analysis in order to examine the relationships between instantiations and strategies further. It examines the strategies used by the participants as evidenced by their gazes, actions, utterances and sketches. In particular, this chapter attempts to explain why learners choose and change their strategies.

Five main explanations of differences in strategies are presented. These are consistent with the quantitative claims made in the previous chapter. Results concerning the differences in representation-specific strategies, variations of strategies for expressing inferences, the connection between interactivity and imagery, the differences in attention paid to parts of external representations, and some difficulties related to external representation and interactivity are presented. This chapter also illustrates how participants change strategies in solving a given task. The chapter concludes by relating the findings and evidence from the quantitative and qualitative analyses.

7.2 DIFFERENCES IN REPRESENTATION-SPECIFIC STRATEGIES IN THE ROOT TASK BETWEEN INSTANTIATIONS

This section explores differences in strategy use which are related to external representations in the simple task (the Root task). The earlier analysis (section 6.3) showed a difference in the strategies, particularly the algebraic-related ones, used for different instantiations for this task. Therefore, the qualitative analysis was used to address the hypothesis that learners' choice of strategies depend not just on the standard external representations given but also on the instantiation. The analysis of representation-specific strategies presented in this section does reveal that there are some strategies associated only with certain types of representations; and that there are differences in the application of some of these strategies.

7.2.1 MAIN FINDING

It is evident in the simple task that more participants apparently use algebraic-related strategies in Dynamic than in the other two instantiations. This suggests that although participants may use algebraic strategies in one type of instantiation, they do not use the same strategy in another type of instantiation. The application of algebraic-related strategies is different between instantiations. Instantiation also appears to influence which aspects of representations participants focus on. There is also evidence that the capability to input equations to generate many external representations quickly overwhelms participants.

7.2.2 EVIDENCE

First, all the apparent representation-specific strategies identified are presented in the Timeline Visualisations shown in Figure 7-1 to show that algebraic-related strategies were used more in Dynamic than in Static and in Interactive. There is a difference in the number of participants using algebraic-related strategies between instantiations. Also, the Timeline analysis shows that graphic-related strategies are apparent in all the three types of instantiations. Roughly the same number of participants used graphic-related strategies across instantiations.

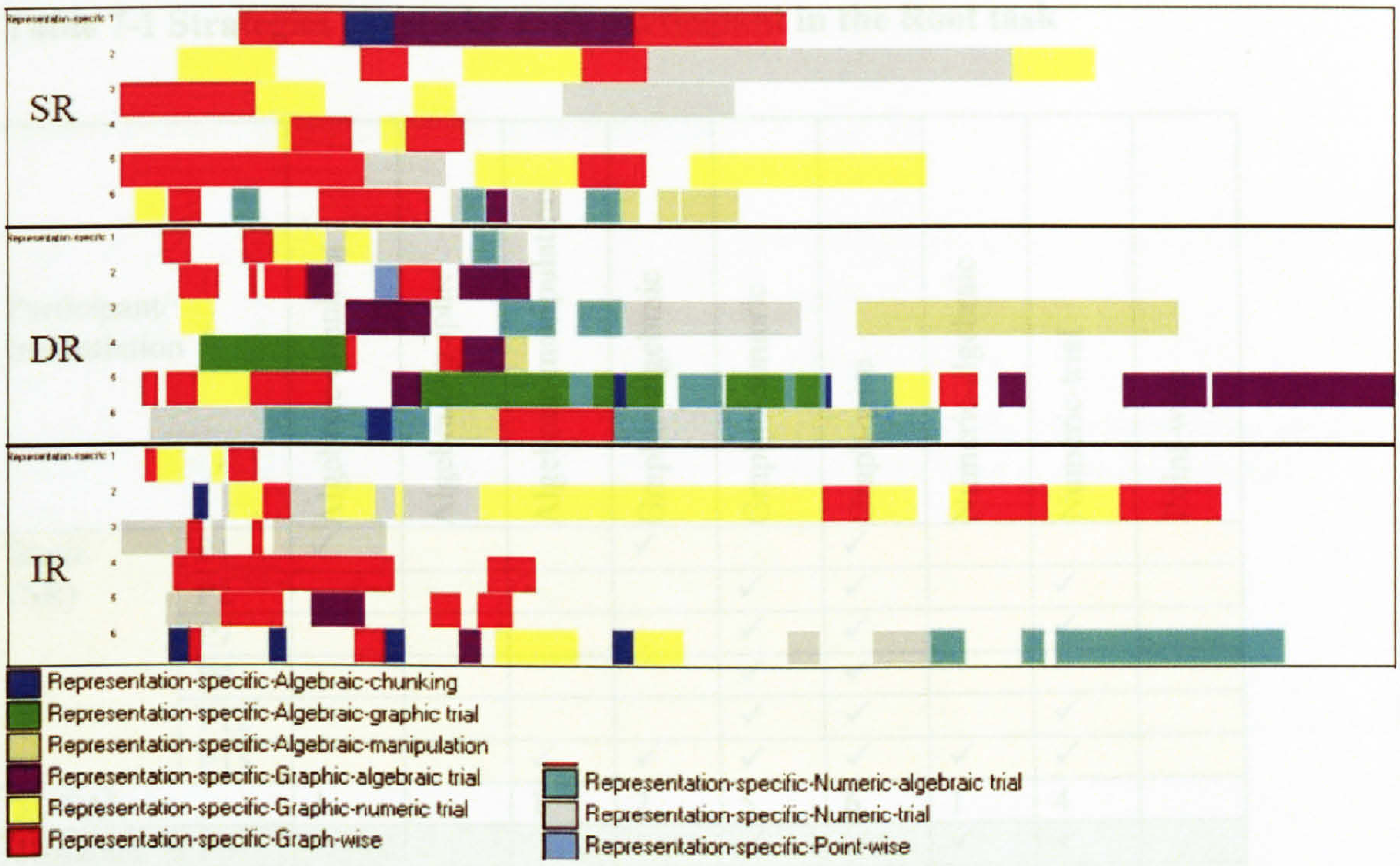


Figure 7-1 Representation-specific strategies under Root task grouped by instantiation

Table 7-1 shows the representation-specific strategies chosen by each of the participants in the Root task between instantiations. This easily shows that all of the participants in Dynamic used at least one algebraic-related strategy. The number of participants who used algebraic-related strategies varies between instantiations, although the numbers of participants are too small to establish statistical significance.

Then, the integrated video data show that algebraic-related strategies were applied differently depending on the type of instantiation. The data also show that there is no apparent difference in the integrated analysis of gazes, actions, utterances and sketches in using graphic-related strategies across instantiations (see Appendix D). In contrast, there is a difference in the actions performed in using numeric-related strategies apparent across instantiations.

Table 7-1 Strategies chosen by each participant in the Root task

Participant/ Instantiation		Algebraic-chunking	Algebraic-graphic	Algebraic-manipulation	Graphic-algebraic	Graphic-numeric	Graph-wise	Numeric-algebraic	Numeric-trial	Point-wise
Static (SR)	P1	✓			✓		✓			
	P2					✓	✓		✓	
	P7					✓	✓		✓	
	P9					✓	✓			
	P12					✓	✓		✓	
	P15			✓	✓	✓	✓	✓	✓	
TOTAL→		1		1	2	5	6	1	4	
Dynamic (DR)	P3					✓	✓	✓	✓	
	P4				✓		✓			✓
	P8			✓	✓	✓		✓	✓	
	P17	✓		✓			✓	✓	✓	
	P16	✓	✓		✓	✓	✓	✓		
	P10		✓	✓	✓		✓			
TOTAL→		2	2	3	4	3	5	4	3	1
Interactive (IR)	P5					✓	✓			
	P6	✓				✓	✓		✓	
	P11						✓		✓	
	P13						✓			
	P14				✓		✓		✓	
	P18	✓			✓	✓	✓	✓	✓	
TOTAL→		2			2	3	6	1	4	
		↑	↑	↑	↑			↑		
Legend		↑ - Algebraic-related strategies SR – Static Root task DR – Dynamic Root task IR – Interactive Root task								

Numeric-trial (appeared in all instantiations)

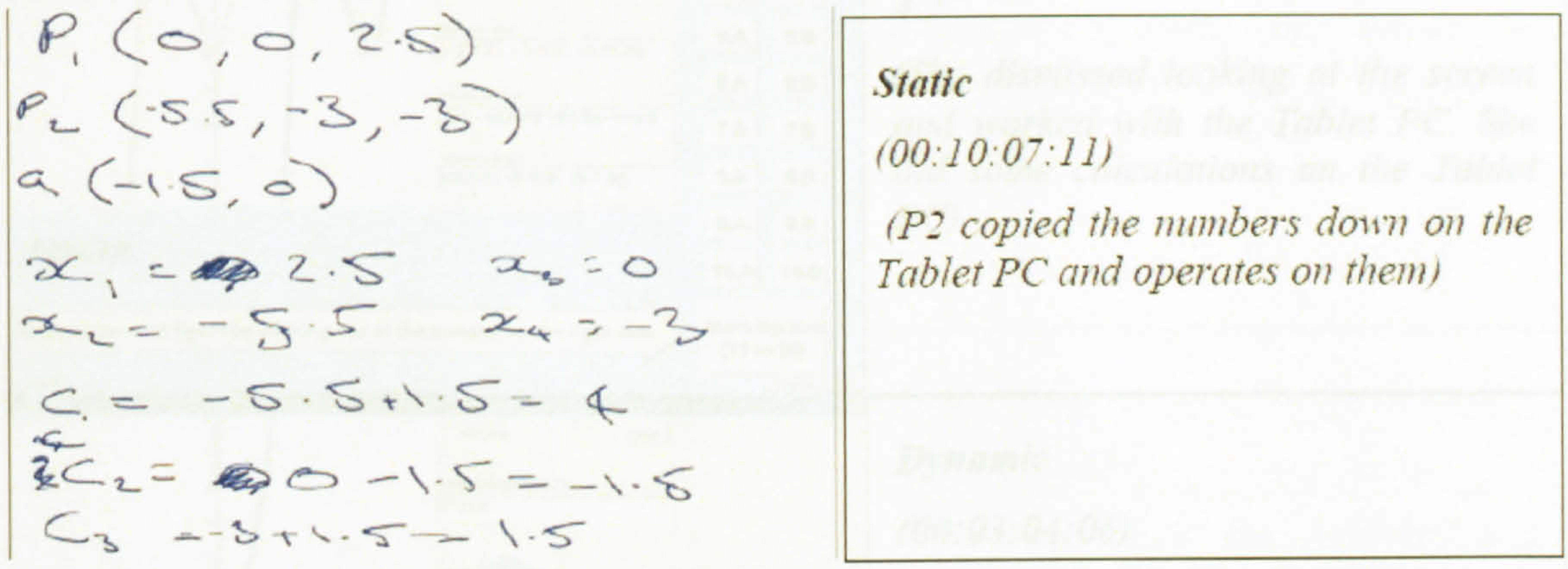
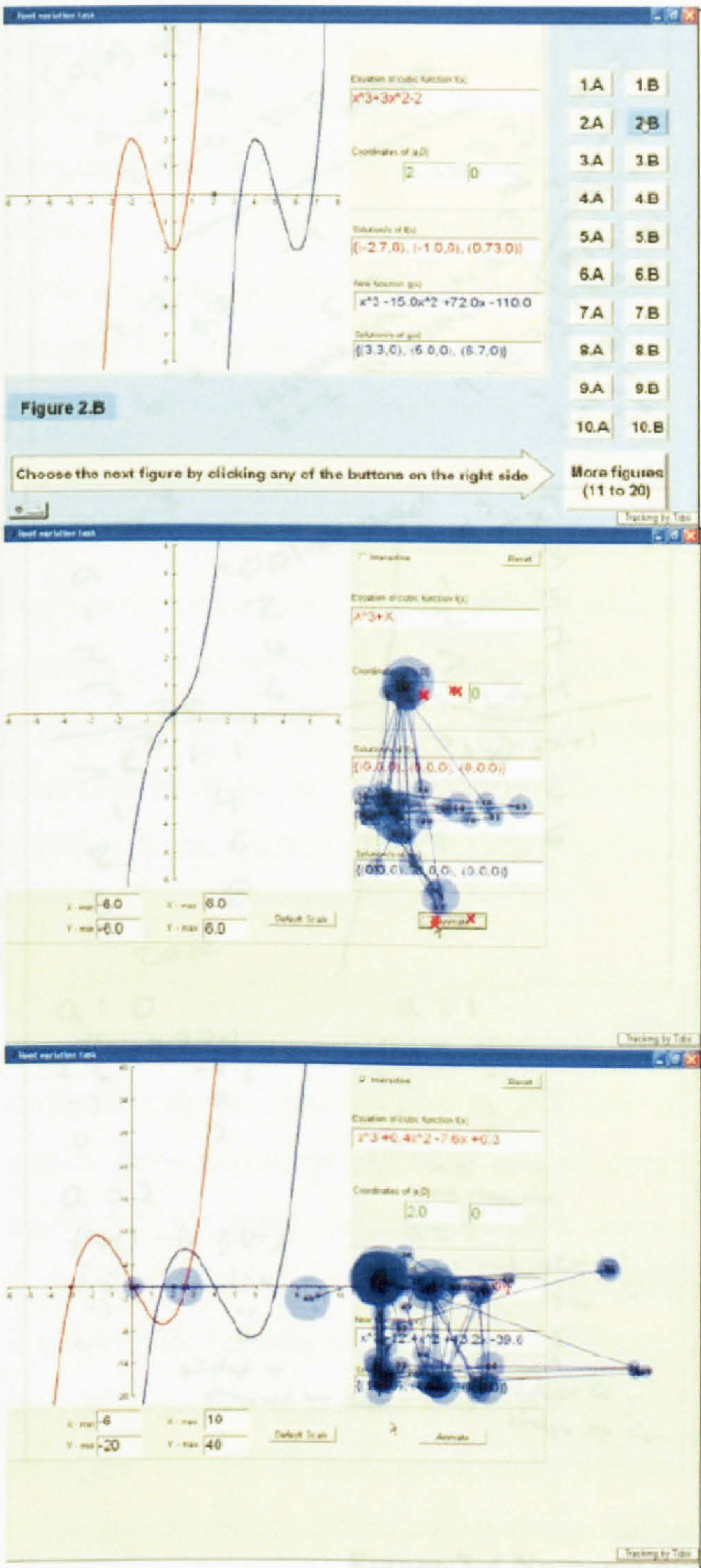


Figure 7-2 A Numeric-trial strategy in Static

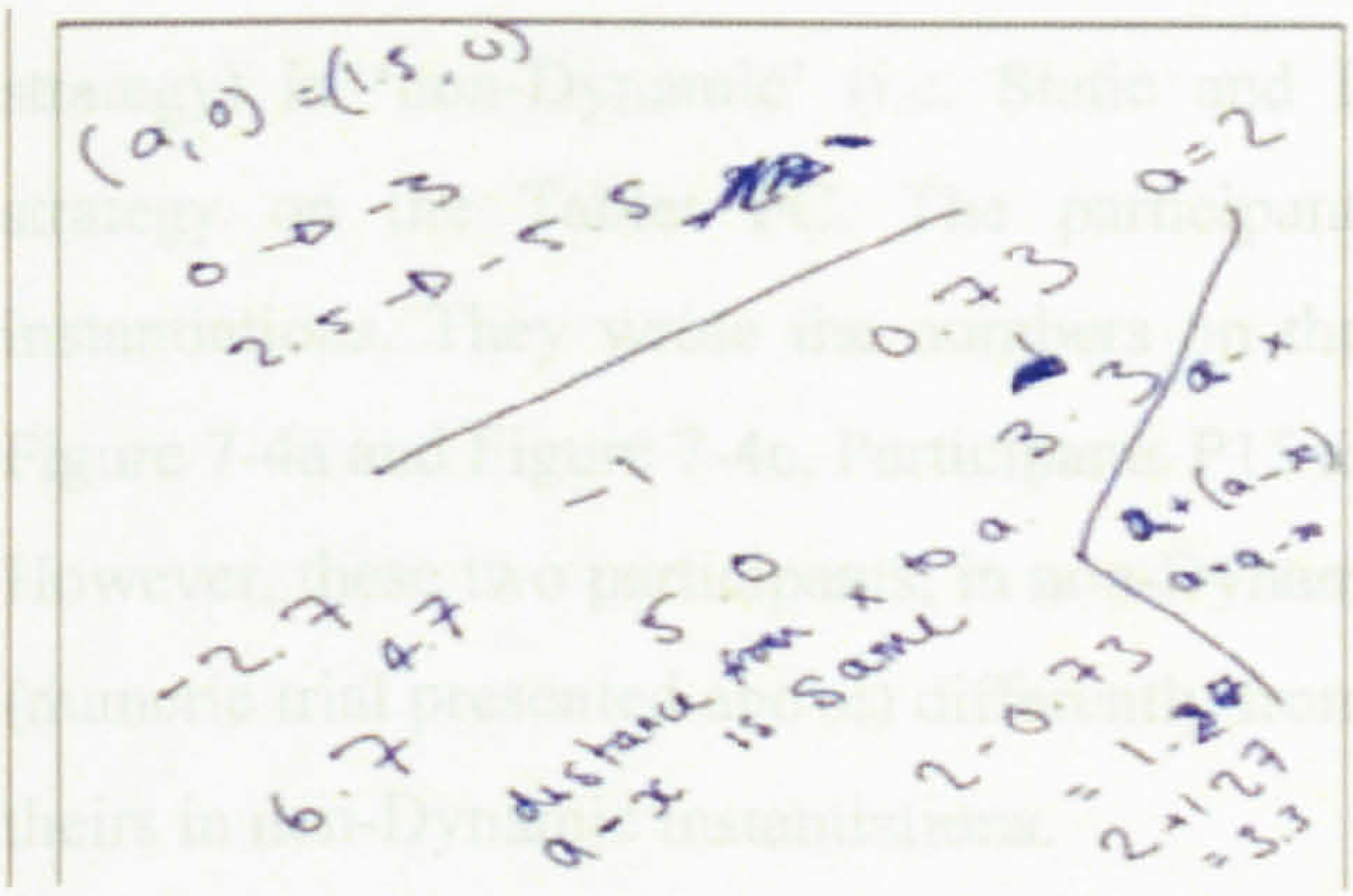
The numeric-trial strategy is characterised by participants' application of some arithmetic operations on the numbers or by their comparison of numeric values. The participants operated on and compared numbers differently between Dynamic and non-Dynamic instantiations. In non-Dynamic, two observations were apparent: the participants just copied the numbers down, then directly applied numerical computations (Figure 7-2) or copied the numbers in a tabular form before operating on them. (Note that the participants who copied the numbers in a tabular form are the participants mentioned in the numeric-algebraic strategy below). These similarities in the use of numeric-trial strategy between Static and Interactive were not apparent in Dynamic.



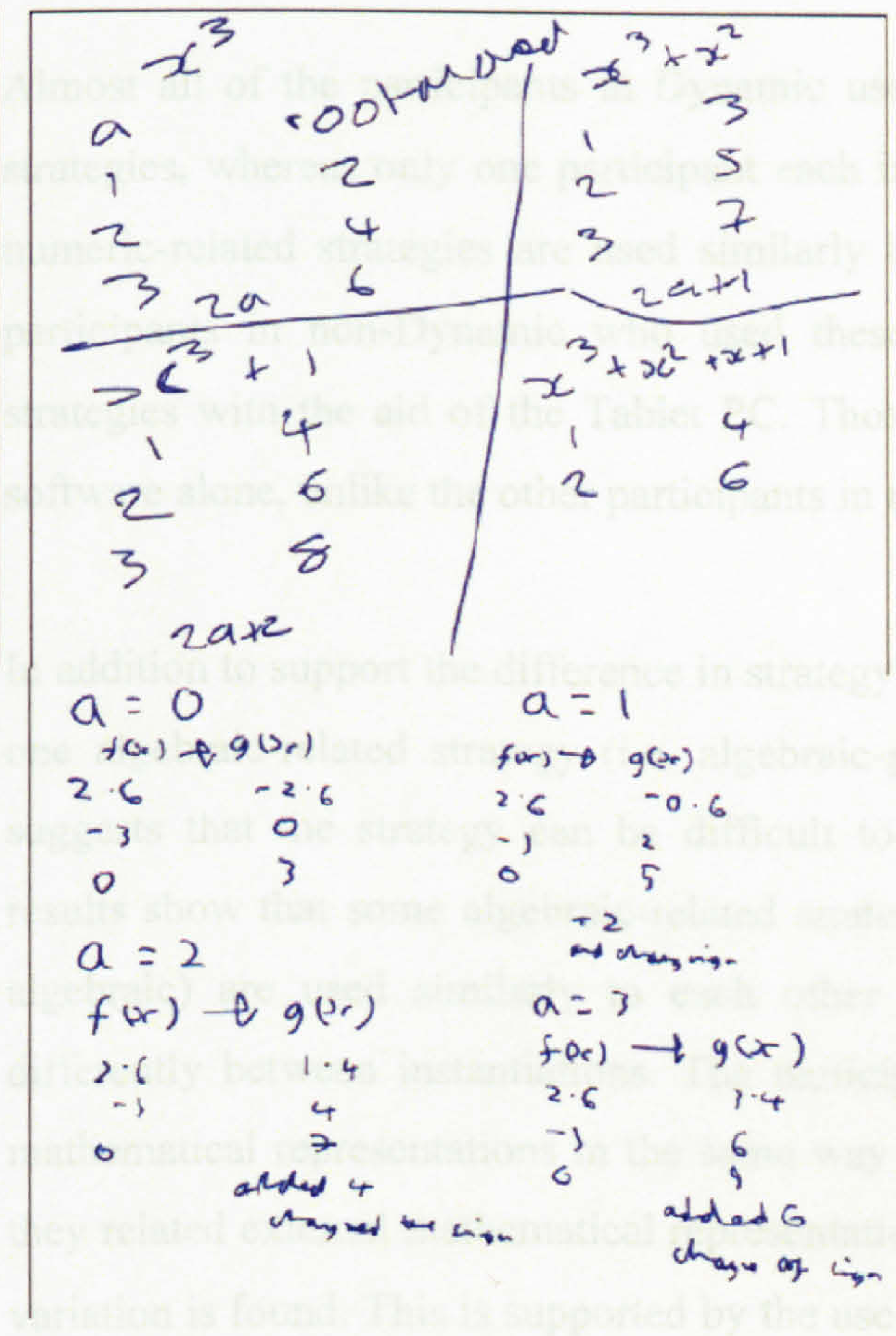
Static *00:03:45:11* *P15: The distance from plus to here is 4.7* *(She dismissed looking at the screen and worked with the Tablet PC. She did some calculations on the Tablet PC)*
Dynamic *(00:03:04:06)* *P17: 1.5 again it is twice the x coordinate of the... Same seems to apply...* *(He entered a number of values for (a, 0) one at a time and looked at the solutions of g(x))*
Interactive *(00:18:49:03)* *P18: I'm looking at the Tablet and the numbers I've written down there so it is more clear... Right... it appears to have added four... that one change in sign and added four...* *(He looked at the numbers on the Tablet PC and the numbers on the screen)*

Figure 7-3 Numeric-trial strategy

In Dynamic (the example given in Figure 7-3b), the participants were observed entering numeric values one-at-a-time and then looking at the corresponding resulting values. Then, they made inferences after looking at each numerical value they entered.



a) Static
(00:05:16:21)
(P15 looked at the numbers on the Tablet PC. Then, she wrote an algebraic expression.)



b) Dynamic
(00:14:25:24)
(P16 looked at the numbers and wrote an algebraic expression)

c) Interactive
(00:20:00:17)
(P18 looked at the Tablet PC and made an algebraic rule in words)

Figure 7-4 Numeric-algebraic strategy

Numeric-algebraic (appeared in all instantiations)

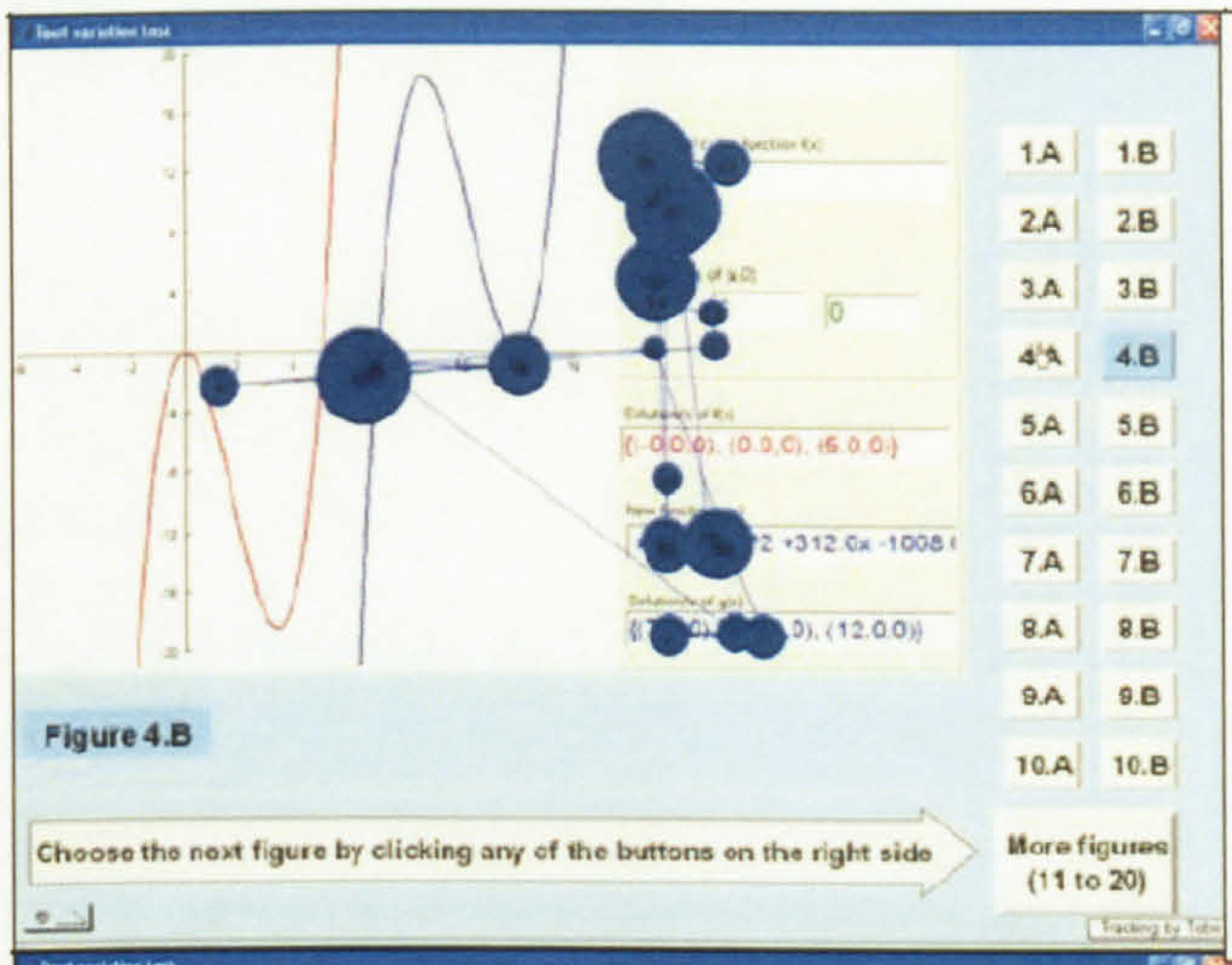
Chapter 6 reported a noticeable difference in the number of participants using the numeric-algebraic strategy between instantiations. Four participants used this strategy in Dynamic. There are also two participants who used this strategy (numeric-algebraic strategy) in 'non-Dynamic' (i.e. Static and Interactive) but were observed using the strategy on the Tablet PC. The participants used this strategy similarly between instantiations. They wrote the numbers on the Tablet PC in a 'table-like' manner (see Figure 7-4a and Figure 7-4c, Participants P15 and P18) and then created an algebraic rule. However, these two participants, in non-Dynamic, used the other numeric-related strategy (numeric trial presented above) differently from others receiving the same instantiation as theirs in non-Dynamic instantiations.

Almost all of the participants in Dynamic use at least one of the two numeric-related strategies, whereas only one participant each in Static and Interactive did so. These two numeric-related strategies are used similarly between instantiations. However, the two participants in non-Dynamic who used these two strategies are observed using the strategies with the aid of the Tablet PC. Those two participants did not work with the software alone, unlike the other participants in non-Dynamic.

In addition to support the difference in strategy use between instantiations, the analysis of one algebraic-related strategy (i.e. algebraic-graphic) that only appeared in Dynamic, suggests that the strategy can be difficult to use in non-Dynamic instantiations. The results show that some algebraic-related strategies (i.e. algebraic-chunking and graphic-algebraic) are used similarly to each other when the instantiation is the same but differently between instantiations. The participants tend to relate the different external mathematical representations in the same way within each instantiation. When the ways they related external mathematical representations are compared between instantiations, a variation is found. This is supported by the use of another two algebraic-related strategies (i.e. algebraic-manipulation and numeric-algebraic strategies).

Further detailed evidence is presented below.

The participants who used the algebraic-chunking and the graphic-algebraic strategies within each instantiation were found to be consistent in terms of: how they related equations to a graph, and the part of the algebraic representations they focused at. The data show that participants seem to link the graph to the overall form of equations (in Static: the change from 'simple' to 'complex' equation was linked to the graph); to the numerical parts of equations (in Dynamic: the movement of the two graphs was related to the change in the coefficients); and to the 'signs' of the parts of equations (in Interactive: the change in the 'signs' of the terms of two equations was also related to the change in the graph).

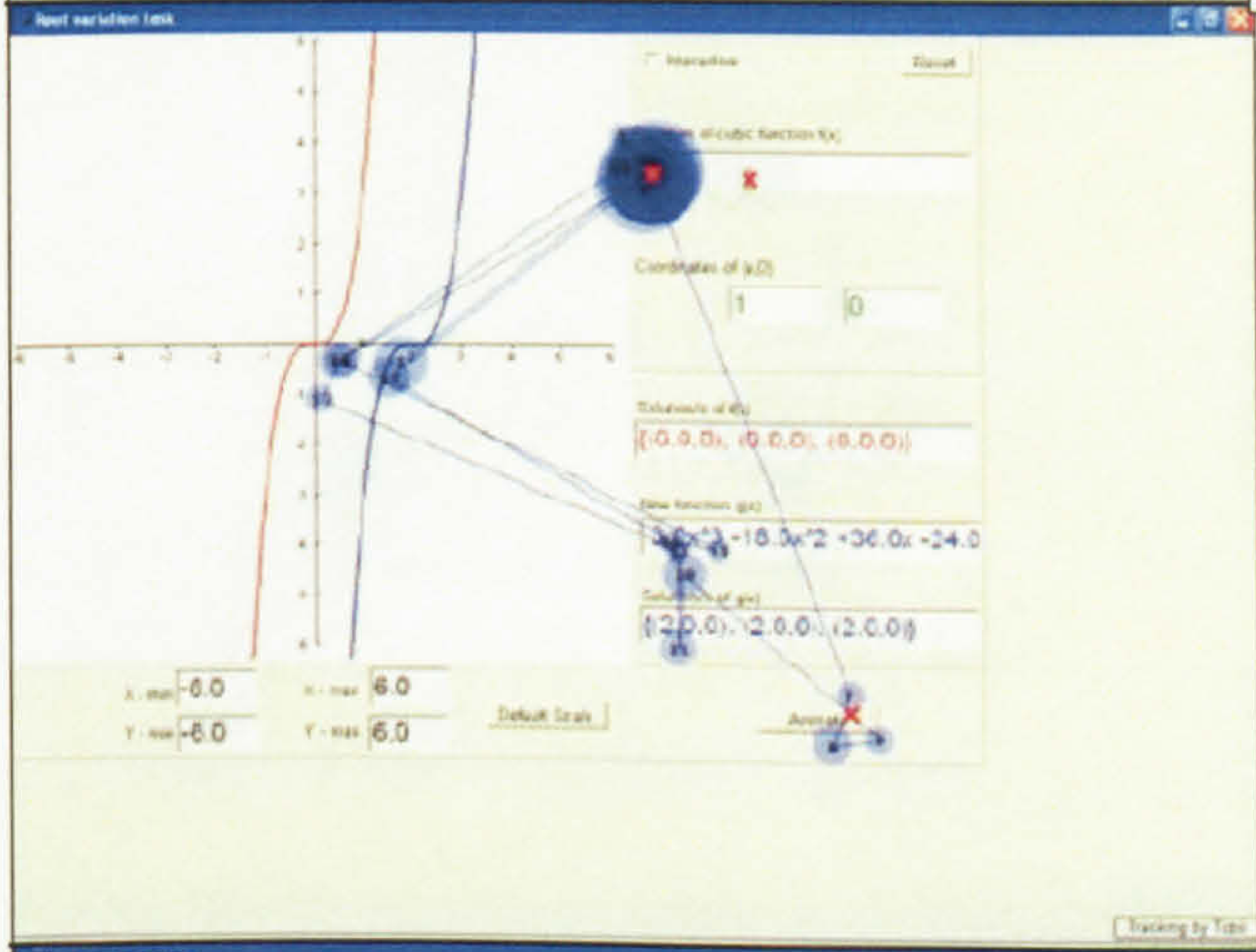


a) Static

(00:03:04:15)

P1: First function is quite simple but when... when it's been rotated you get quite complex

(P1 looked at two pictures and compared the two equations)

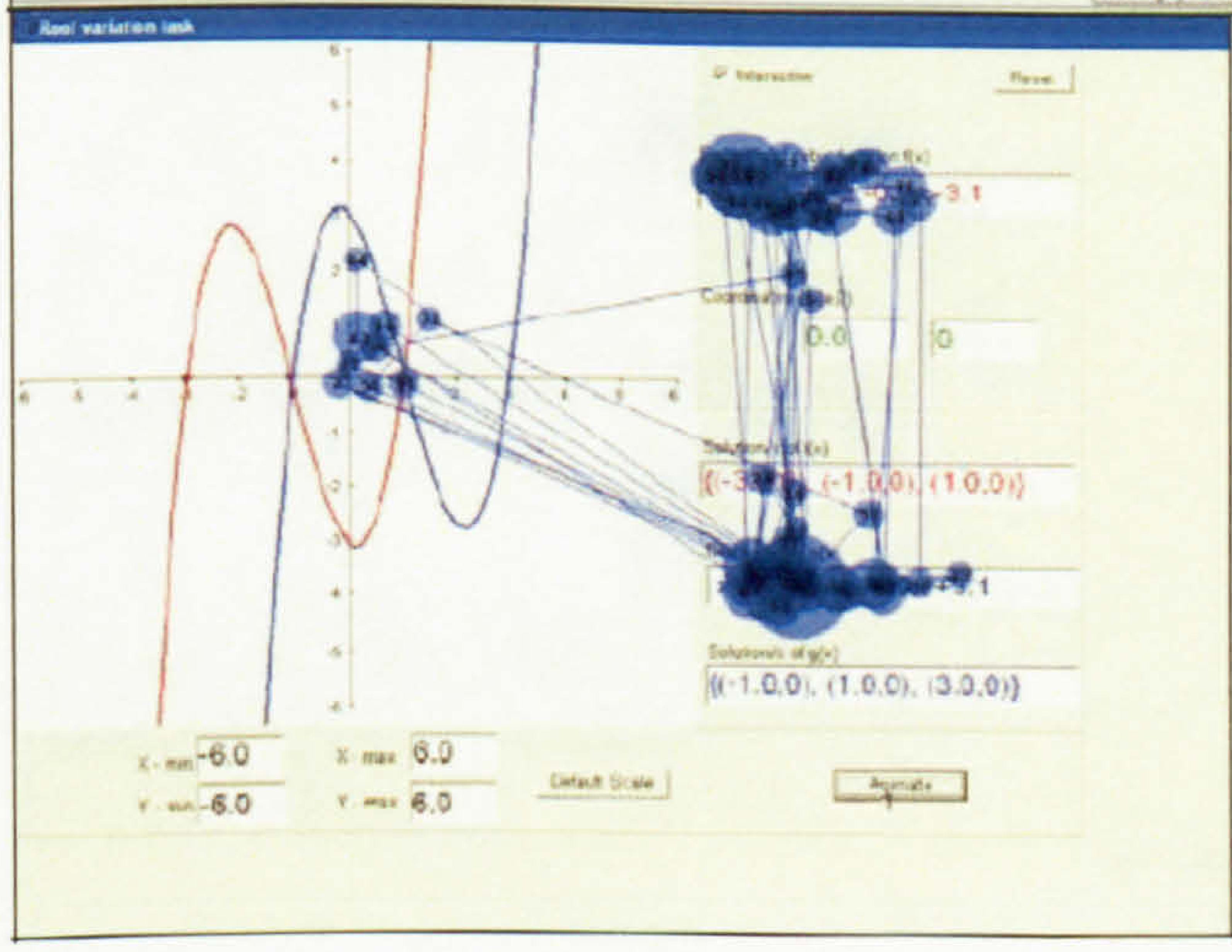


b) Dynamic

(00:10:47:09)

P16: Which is the same... for that... no matter what the coefficient is... for this case

(P16 changed the coefficient of X^3 twice and look at the solutions)



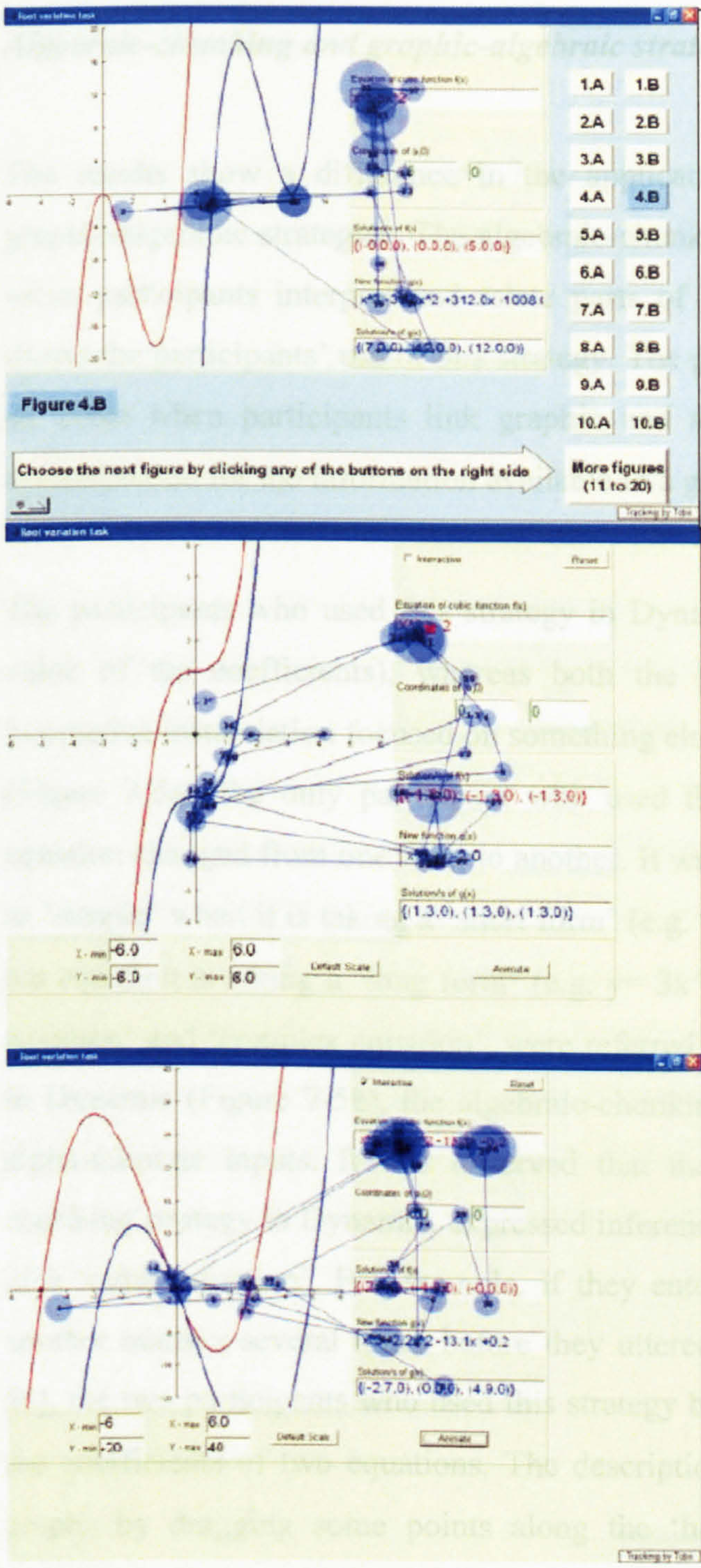
c) Interactive

(00:01:36:02)

P6: Err... Looking at the old function and now looking at the actual equation I mean the new function it's the same... apart from the fact that the sign has changed from the actual value...

(P6 looked at the equations and compared the signs of each term)

Figure 7-5 A comparison of the way an Algebraic-chunking strategy was used



a) Static

(00:03:33:13)

P1: So the solutions are quite simple although they are the same but the other side of the point of rotation but when you rotate... the equation becomes complex

(P1 looked at two pictures and then gazed at the solutions on the graph. Then, she compared the two equations)

b) Dynamic

(00:04:00:05)

P4: x^3 + 2 and if I rotate around (0, 0)... it goes down this way and that becomes x^3 -2... That's interesting..."

(P4 related the equation with the graph's behaviour)

c) Interactive

(00:07:58:14)

P18: So from this I think I could... every time I rotate the function... this coefficient here will change signs... and the end value will change sign as well...

(P18 looked at the two equations and the graph)

Figure 7-6 A comparison of the way a Graphic- algebraic strategy was used

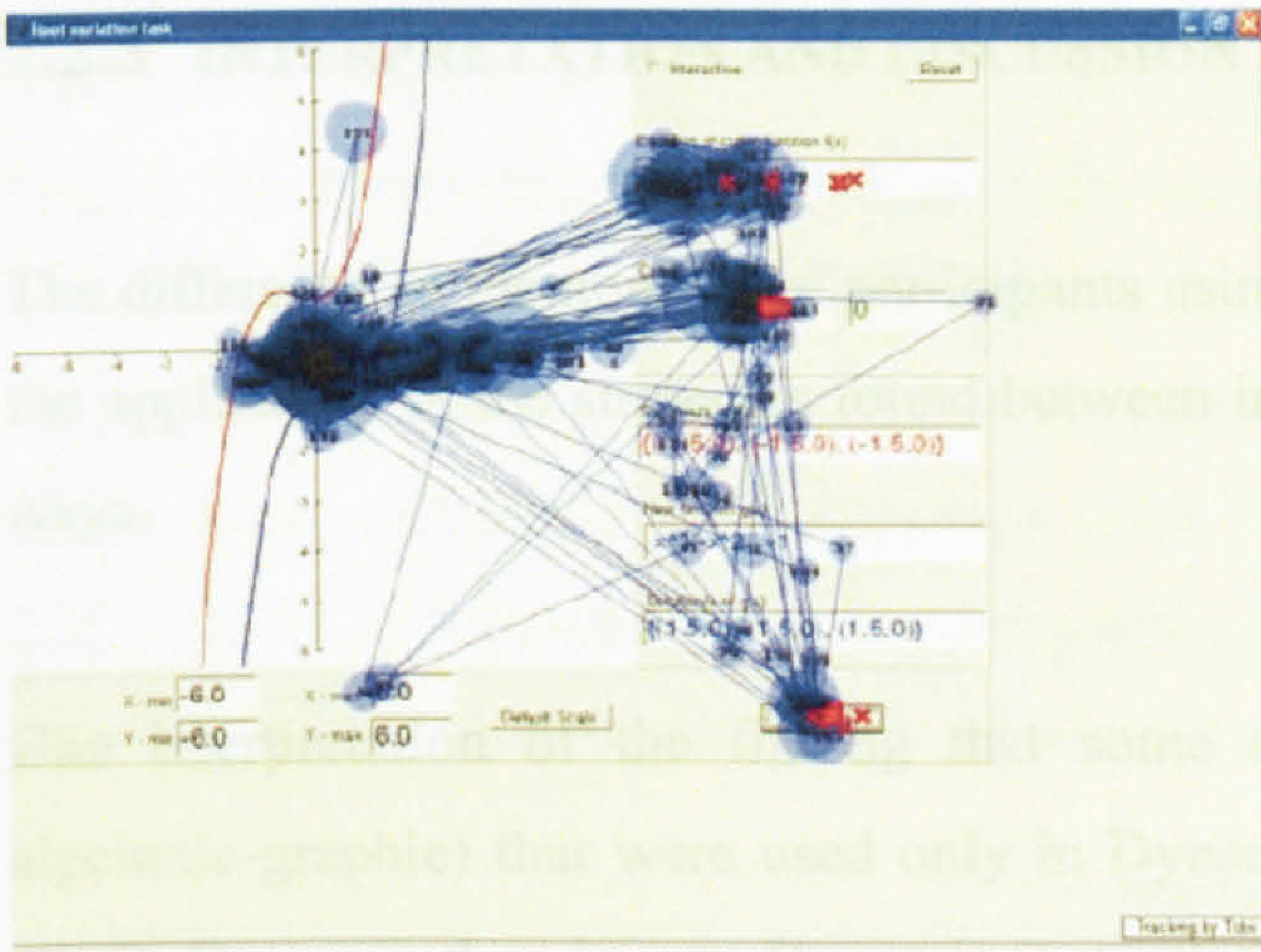
Algebraic-chunking and graphic-algebraic strategies (appeared in all instantiations)

The results show a difference in the application of the algebraic-chunking and the graphic-algebraic strategies. The algebraic-chunking strategy is characterised as an event when participants interpret and relate parts of an algebraic representation. Figure 7-5 shows the participants' use of this strategy. The graphic-algebraic strategy is identified as an event when participants link graphic and algebraic representations to construct a symbolisation for the information available in a graph (Figure 7-6).

The participants who used this strategy in Dynamic focused on one thing (the numeric value of the coefficients), whereas both the participants who used this strategy in Interactive instantiation focused on something else (the signs of the coefficients). In Static (Figure 7-5a), the only participant who used this strategy described how the original equation changed from one form to another. It was apparent that the equation was viewed as 'simple' when it is taking a 'short form' (e.g. y = x^3) and viewed as 'complex' when the equation is taking a 'long form' (e.g. y= 3x^3 – 2x^2 + 3x – 7). The terms, 'simple equation' and 'complex equation', were referred to in similar ways by other participants. In Dynamic (Figure 7-5b), the algebraic-chunking strategy was used by changing some alpha-numeric inputs. It was observed that the participants who used the algebraic-chunking strategy in Dynamic, expressed inferences after changing a coefficient of a term of a 'cubic equation'. For example, if they entered 'y = 4x^3', they changed '4' with another number several times before they uttered an inference. In Interactive (Figure 7-5c), the two participants who used this strategy both described the change in the signs of the coefficients of two equations. The description was given after changing one of the graphs by dragging some points along the 'horizontal-axis'. In all events coded as graphic-algebraic in Interactive (also in algebraic-chunking in Interactive), there was one specific graphical representation that no participant changed (i.e. the graphical point of rotation, which is at the point of origin (0, 0)).

Algebraic-graphic and algebraic-manipulation (mostly appeared in Dynamic)

The results also indicate that some strategies can be difficult to carry out in non-Dynamic instantiations. The algebraic-graphic strategy, for example, is a strategy identified as an event when participants modify equations in the light of the information gained by comparing successive graphs. In all the occurrences of this strategy, the participants used the strategy by changing the equations several times before they made an inference (e.g. Figure 7-7). In their inferences, they related whatever change they noticed in the equation to the graph of the function. Another algebraic-related strategy that did not appear across instantiations was the algebraic-manipulation strategy. This strategy is identified when participants work or operate on equations or algebraic expressions; or when they translate a problem statement to algebraic assignments and equations. Three participants used algebraic-manipulation in Dynamic whilst only one participant used this strategy in Static. In all the events coded, the participants were observed using the strategy by manipulating algebraic equations/expressions on the Tablet PC. In the examples given, one was operating on symbols (Figure 7-8a) whilst in the other example, manipulations of algebra consisted of text (Figure 7-8b). It appears that the algebraic-manipulation strategy is used in similar ways by the three participants in Dynamic and the participant in Static. The strategy was used with the aid of the Tablet PC. Also, it is important to note that the participant who used this strategy spent more time working with the Tablet PC than using the software in Static.



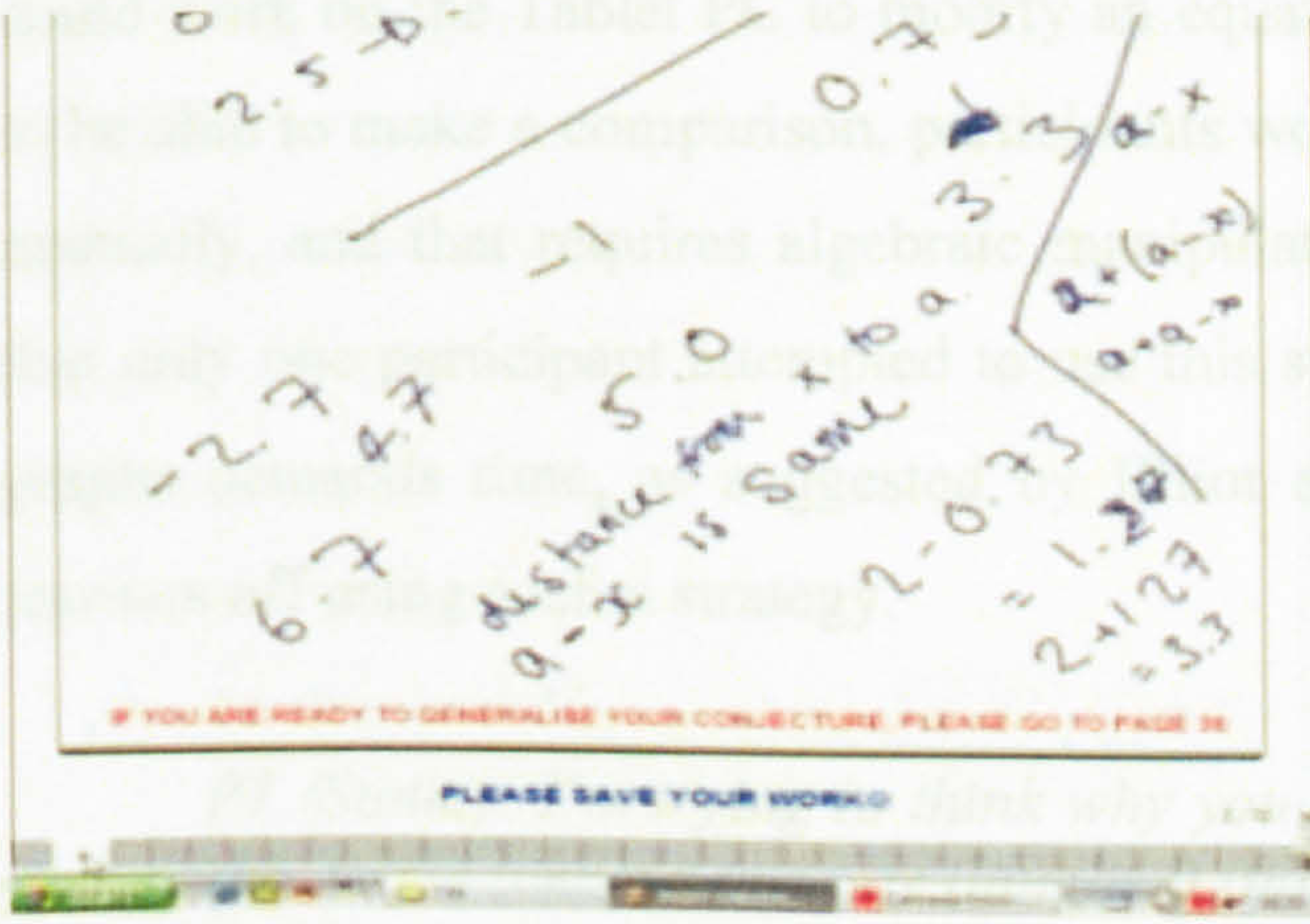
Dynamic

(00:03:45:11)

P10: It doesn't matter which cubic function I put in... Any cubic function... if I rotate it by (0, 0) one of the roots will always be the same...

(P10 inputted several equations and looked at the graphical solutions)

Figure 7-7 An Algebraic-graphic strategy by P10

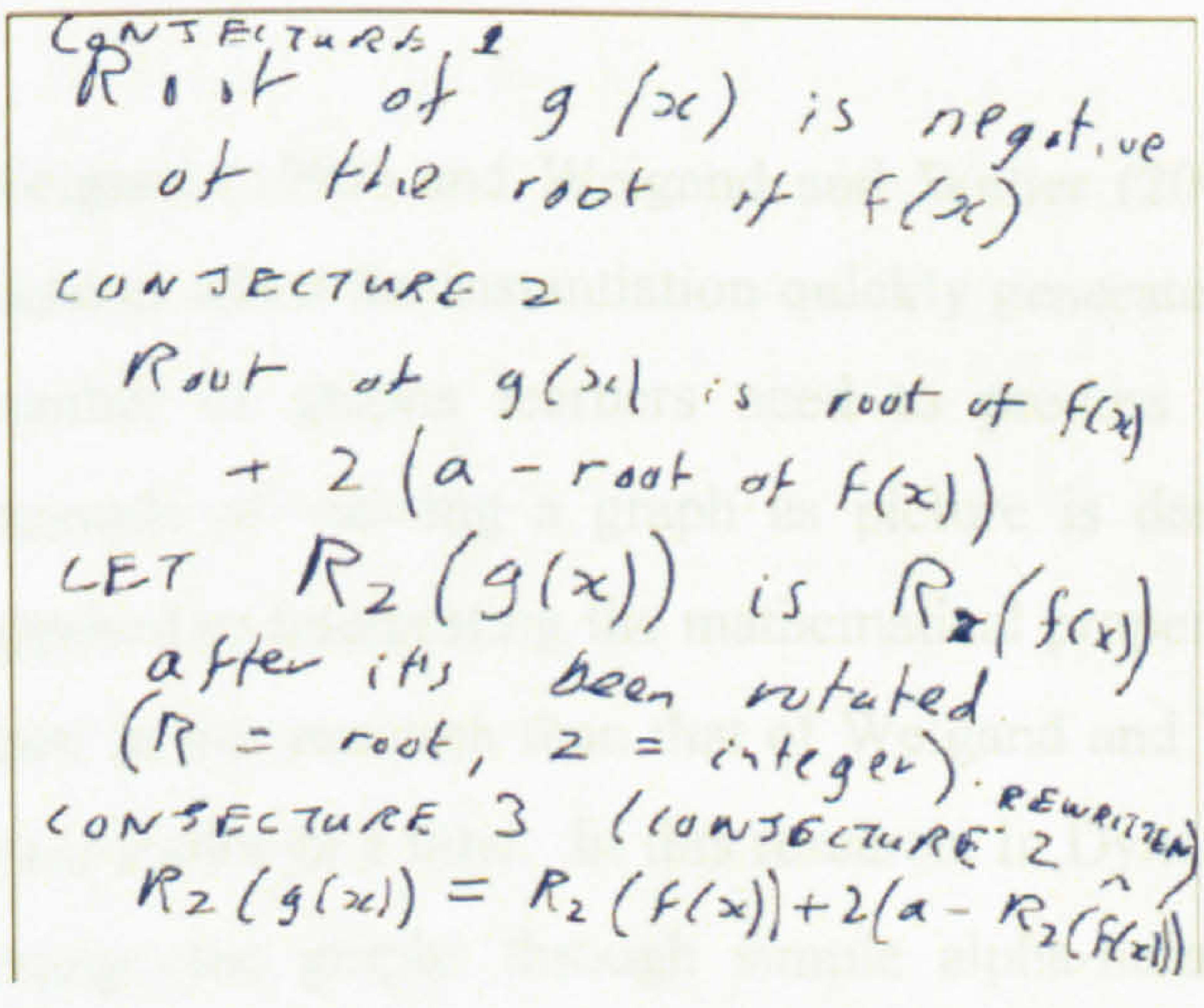


a) Static

(00:05:40:14)

P15: (wrote)

(P15 wrote on the Tablet PC trying to come up with an algebraic rule)



b) Dynamic

(00:16:26:13)

P8: So this is... the root... and...

(P8 wrote on the Tablet PC and attempted to come up with an algebraic rule)

Figure 7-8 A comparison of the way an Algebraic-manipulation strategy was used

7.2.3 INTERPRETATION AND DISCUSSION

The difference in the number of participants using certain strategies and the difference in the application of the strategies found between instantiations can be explained in various ways.

One interpretation of the finding that some of the algebraic-related strategies (e.g. algebraic-graphic) that were used only in Dynamic are that such strategies are easier to use in Dynamic than in non-Dynamic instantiations. For example, the algebraic-graphic strategy can be used in non-Dynamic instantiations. However, to do so requires participants to ignore the representation provided and construct their own. Participants could work on the Tablet PC to modify an equation and then graph it manually. In order to be able to make a comparison, participants would need to construct at least two graphs manually, and that requires algebraic manipulation. As presented above, the data show that only one participant attempted to use this strategy in Static. Manual construction of graphs demands time, as suggested by Elliot et al. (2000), and this overhead can put learners off using such a strategy.

> *P1 (Static): I'm trying to think why you get that solution when you rotated it but that might take me one day (laughed) (00:05:51:00)*

Weigand (1991) and Weigand and Weller (2001) claim that learners view graphs like pictures when the instantiation quickly generate many graphs in a short time because the number of graphs learners need to process cognitively is overwhelming them. An example of viewing a graph as picture is describing it as a shape like 'rounded' as opposed to interpreting the mathematical properties of a graph. There are more equations used in this research than that of Weigand and Weller's studies where only one equation were shown at a time. In this research, in Dynamic instantiation, participants can quickly change the graphs through simple alpha-numeric input. The external representations included changes in two equations and also numerical representations. Watching the videos revealed that participants also experienced difficulty in interpreting the equations. Since it is easy to change and generate graphs and equations in this type of instantiation, participants tended to use algebraic-related strategies. However, the findings above suggest that participants also seemed to view equations in a pictorial way. Their utterances suggest that when the equation is short (in terms of the amount of text displayed in relation to the width of the rectangular text field in which an equation can be typed) many referred to the equation as being 'simple'; and when the equation is long, they referred to it as 'complex'. In other words, each equation was interpreted in terms of how long it looked, as opposed to in terms of its mathematical properties.

In Dynamic, participants focused on the numerical coefficients of the equations and linked them to the graph. This is perhaps related to the actions required in Dynamic instantiation. This sequence of actions involved inputting equations and the numbers, and then seeing the graphical representations generated. The instantiation can therefore show the effect of the change of one numerical input on the graph. So for example, in Figure 7-6 in Dynamic, P4 entered an equation x^3 + 2 and a point of rotation (0, 0). The software shows both the resulting rotated graph and the corresponding equation x^3 – 2. P4 then related the change in the graph to the change in the numbers in the equations.

> *P4 (Dynamic): x^3 + 2 and if I rotate around (0, 0)... it goes down this way and that becomes x^3 – 2 ... That's interesting... (00:04:00:05)*

This is also supported by the participants' actions in using a numeric-trial strategy. Because the software can return resulting numerical representations based on an input, participants used the software as an aid in doing their mental calculations.

> *P3: I will try different values for the coordinates... that gives me similar... similar effect but the solutions are bigger... it seems to be doubled... adding on... seems to double the x coordinate (00:07:18:06)*

A particular cognitive benefit of the Dynamic instantiation compared to the Static instantiation, according to Rogers and Scaife (1996), is its capability of 'dyna-linking' the process of linking and manipulating representations at the interface, which helps learners to visualise at different levels of abstraction. What seems to be happening is that participants relate their alphanumeric input to the other representations. However, a different effect of dyna-linking was found when participants can manipulate the graphs and relate it to algebraic representations. This was observed in the Interactive instantiation.

In Interactive, the results show that the participants noted changes in the signs of the coefficients in two equations and related these changes to a graphical representation. In this particular software, the required action is different from Static and Dynamic. The software has initial representations to start with: an equation, a numerical coordinate and an equivalent graph of this equation. They can change the graph and the numerical coordinate, however, the participants were found to pay particular attention to the default display.

> *P6 (Interactive): Err... Looking at the old function and now looking at the actual equation I mean the new function it's the same aside from the fact that the sign has changed from the actual value... (00:01:36:02)*

Because the initial rotation point was set to be the origin, the effect of the rotation is to change the sign. Rather than focusing on the absolute distances of the graphical coordinates, which was found happening in Static, participants focused on the numerical coefficients in Interactive. Had the default rotation point been elsewhere, the effect of the rotation would have been other than the change in sign. This indicates that the position of graphical points, in this particular case a distinct point location and as an initial location of a 'draggable' representation, can influence learners' strategy. This corroborates the findings of Dunham and Osborne (1991) that learners have a tendency to focus on distinct points. Also, instead of looking at the representations in a pictorial way (Weigand, 1991, Weigand and Weller, 2001), having to manipulate graphs and to notice simultaneous change to other forms of representation can draw their attention to other representations. This is somewhat similar to a split-attention effect (Mayer and Moreno, 1998).

There were some participants who used algebraic-related strategies in Static and in Interactive instantiations. However, these participants dismissed working on the computer and used algebraic-related strategies with the aid of the Tablet PC. The facility of manipulating equations is difficult to do in Static and Interactive instantiations. This lends support the finding of Weigand (1991) and Weigand and Weller (2001) that learners with preference for algebraic representations may not feel comfortable working with a computer. Thus, they preferred to work on the Tablet PC (Pirie, 1996) rather than using the software alone.

The results above support the hypothesis that learners' choice of strategies depend not just on the standard external representations given but also on the instantiation.

7.3 IMAGERY AND INTERACTIVITY

The hypothesis related to imagery is that externalised mental construction of images varies between instantiations. With the digital approach, it was possible to identify a range of imagining strategies which externalised learners' construction of mental images. The results presented in this section are about the use of imagining strategies based on the complex tasks, Chord and Tangent tasks that predominantly elicited graphic-related strategies since there are only few imagining strategies that occurred in the simple task (the Root task). However, it also illustrates how imagining strategies are used in the simple task.

7.3.1 MAIN FINDING

The analysis shows that the type of instantiation can influence learners' use of imagining strategies. The results presented in the previous chapter suggest that the time participants spend with the task when they are using imagining strategies varies between instantiations. In this chapter, it is found that the time spent, the number of occurrences, and the choice of strategies related to imagining vary between Interactive and non-Interactive instantiations. It is also found that participants appear to apply imagining strategies differently depending on the number of external representations being manipulated. Participants tend to use their eyes or mouse in manipulating one external representation at a time; their hands in manipulating or comparing manipulations of two representations at the same time; and use pen and paper in comparing more than two manipulated representations.

7.3.2 EVIDENCE

The Timeline analysis of imagining strategies of particular participants to show example of their choices between Static and Interactive. This is followed by the presentation of imagining strategies related to gestures, gazes, and sketches.

Occurrences of imagining strategies chosen Chord task

In inspecting the Timeline Visualisation of imagining strategies for every participant, in the Chord and Tangent tasks, the analysis revealed that: the time spent, as indicated by the length of occurrence of the strategy; the number of occurrence; and the use of different types of imagining appear to vary between Interactive and non-Interactive instantiations. For example, Figure 7-9 shows that P16 used three different kinds of imagining strategies in Static whereas only two in Interactive. In addition, there were more imagining strategies that occurred in Static than in Interactive. Also, the length of time, as indicated by the length of the Timelines, seemed shorter in Static than in Interactive. A similar kind of situation can be seen for P8 and P10 (see other Timelines in Appendix F).

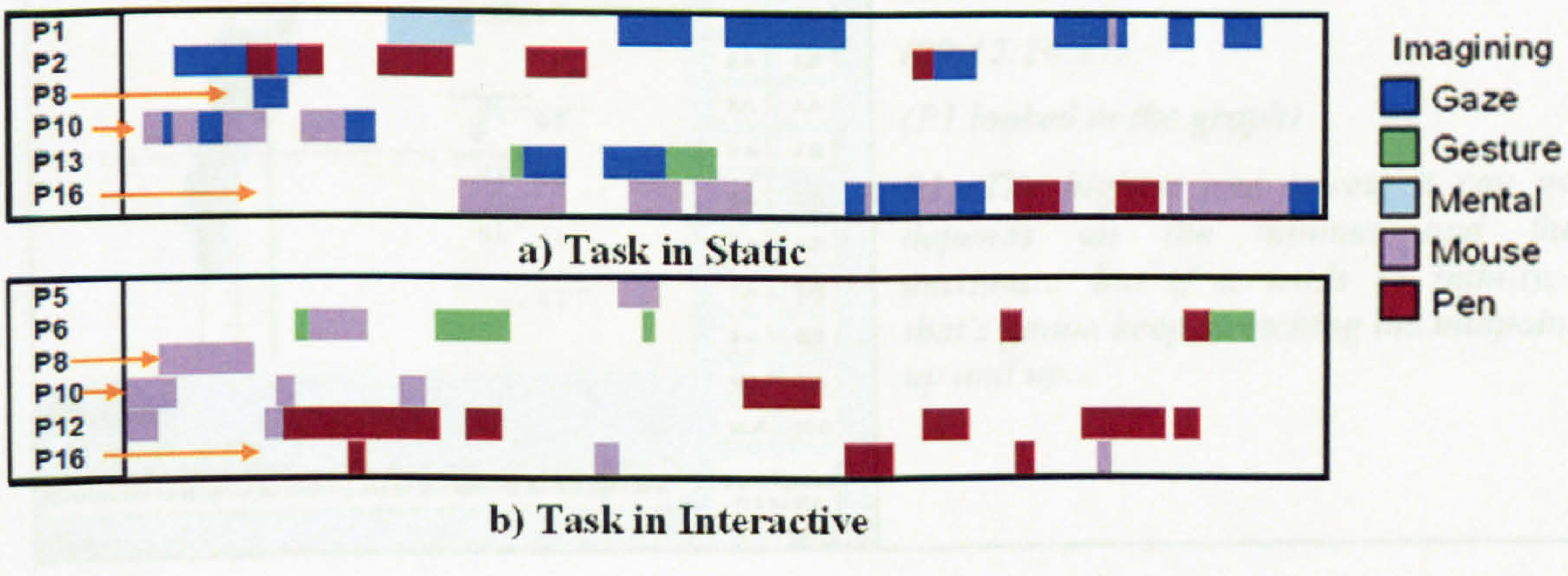


Figure 7-9 A comparison of participants imagining strategies in the complex tasks

Imagining strategies for 'drawing and moving' graphical representations

Using the analysis technique, it was possible to capture some situations when participants visualise. These situations below are examples of participants' use of imagining strategies. The participants were identified using the strategies 'to draw' and 'to move' graphical representations. The graphical representations that they were 'drawing and moving' in Static, also sometimes in Dynamic, are the external representations that they cannot manipulate.

When they move or draw a representation one at a time, they were observed using either gaze-drawing strategy or mouse-drawing strategy. In the two tasks, for non-Interactive instantiations (i.e. Static and Dynamic), participants were found using the gaze-drawing strategy as if drawing lines using their eyes. In watching the gaze video, it was observed that P1's gazes were moving within the contour of the cubic graph. The gaze movement is depicted like lines being drawn from a point along the curve to another point, one by one. As this kind of gaze imagining occurred, participants uttered lines like the ones in the examples (e.g. Figure 7-10). The same situation was found happening in visualising images using the mouse. The mouse pointer was observed being used like lines being drawn (e.g. Figure 7-11).

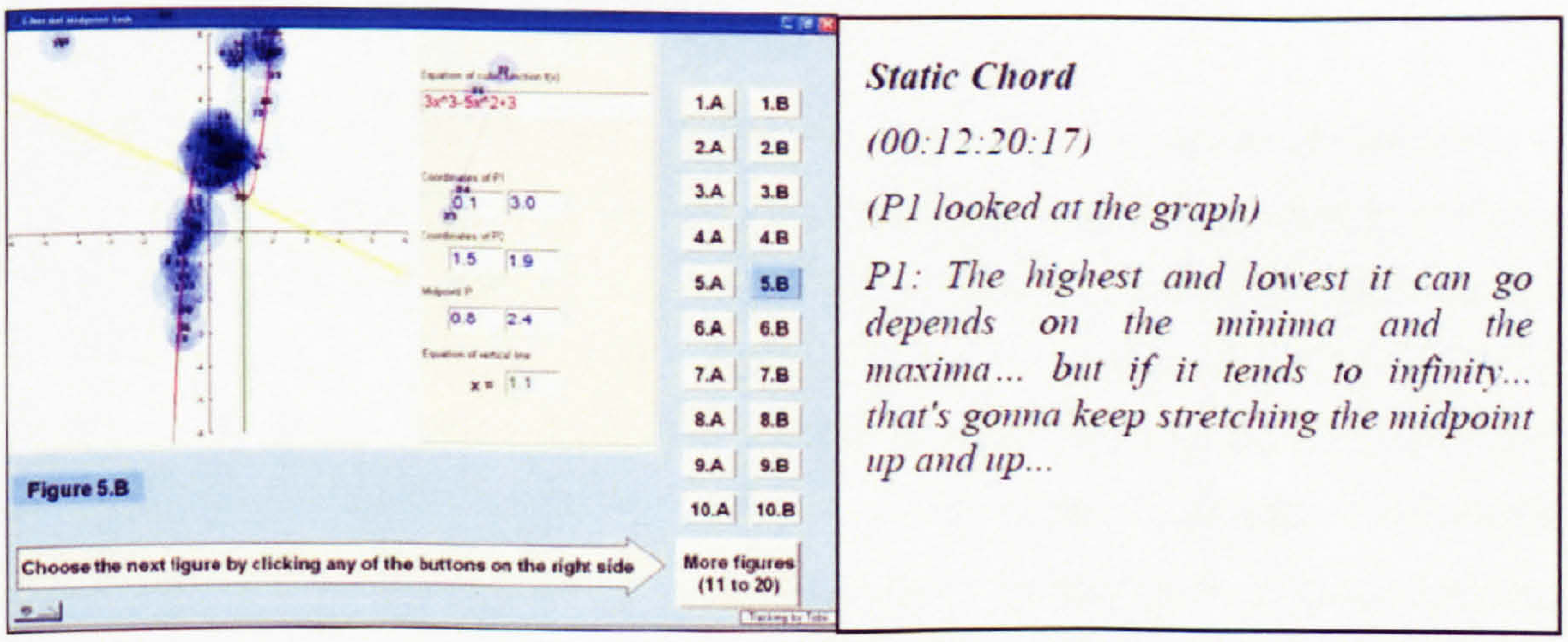


Figure 7-10 A Gaze-drawing strategy

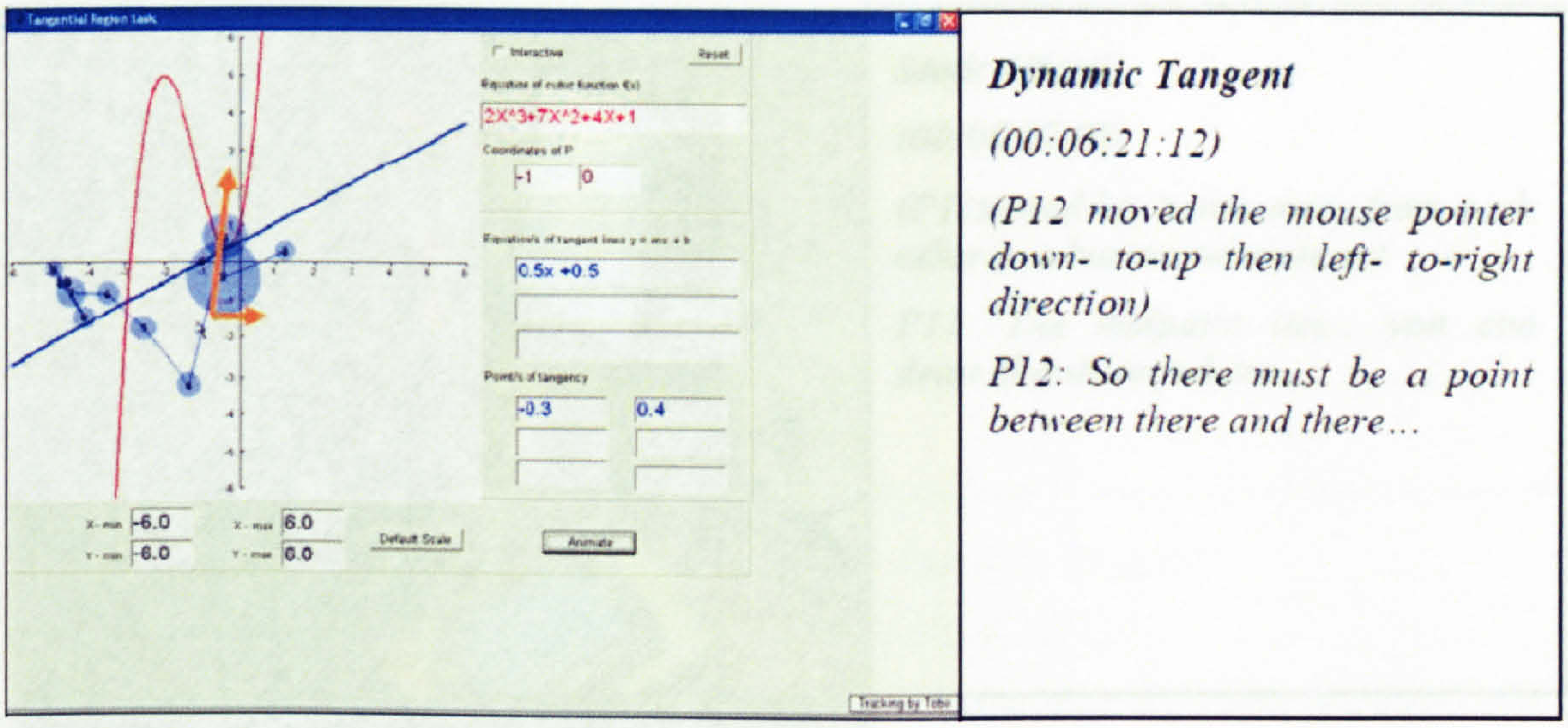
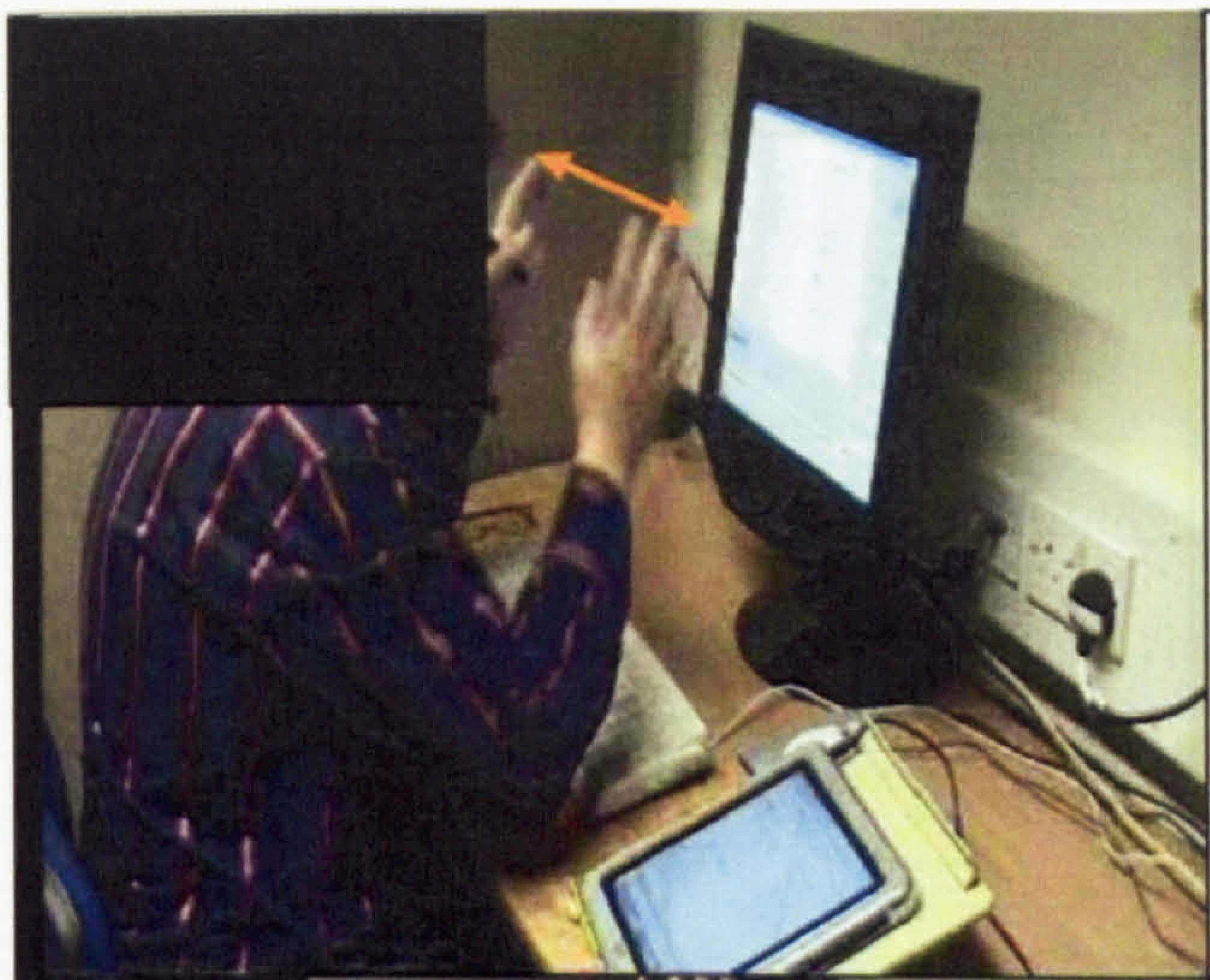


Figure 7-11 A Mouse-drawing strategy

There were situations when they had to move an object and compare it with another object. Some participants were observed using their hand like a 'place-holder', hence, a gesture-drawing strategy. In all the events identified as a gesture-drawing, the participants were observed using the strategy in a similar way. Figure 7-12 shows some typical examples where participants use their hand to gesture a drawing of a line, comparing it with a line or some lines on the screen.

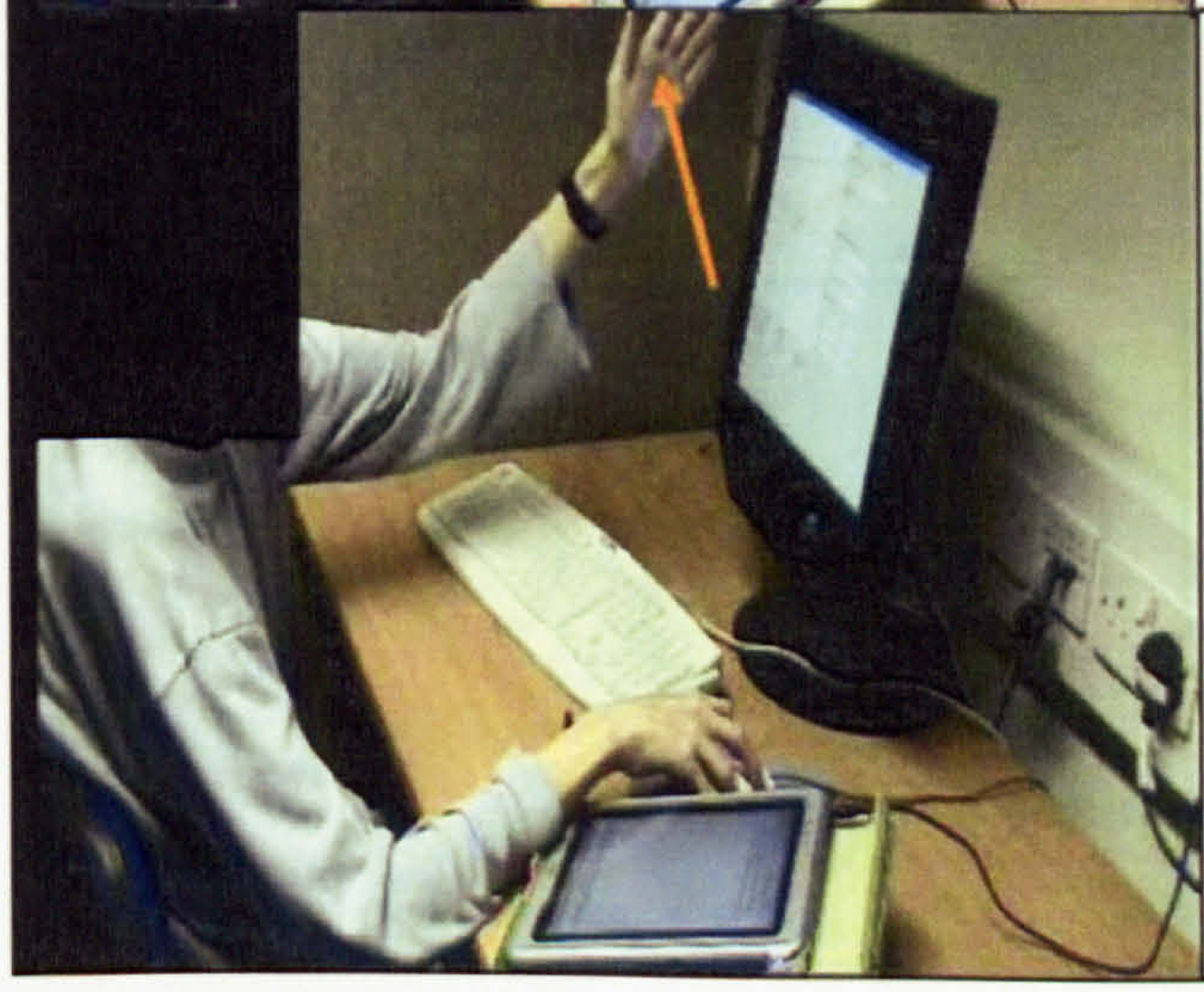


Static Chord

(00:06:57:08)

(P11moved his hands away from each other in a horizontal manner)

P11: The midpoint lies... you can draw chords anywhere...



Dynamic Tangent

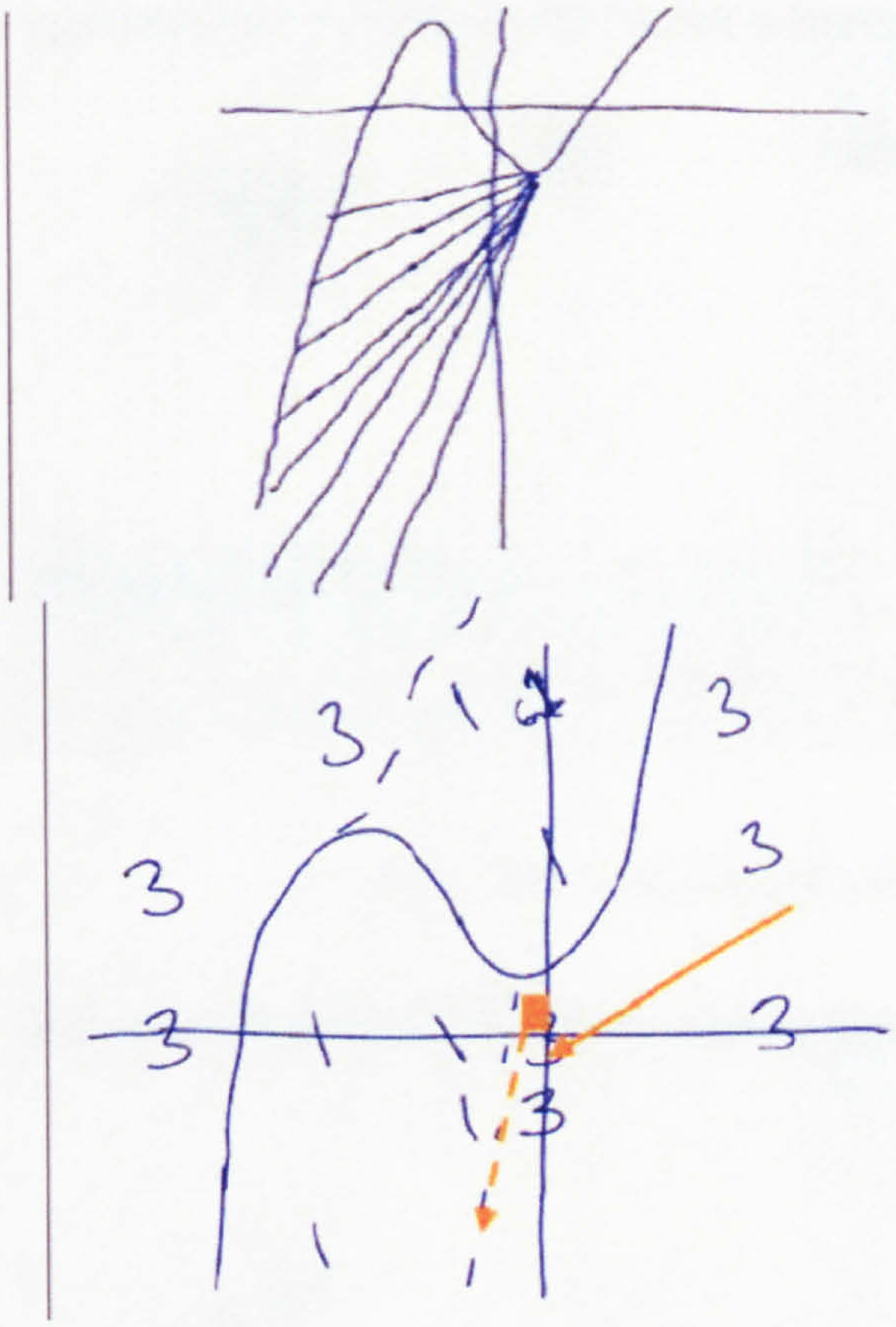
(00:00:44:21)

(P17 moved his hand in an upward manner)

P17: When that tangent crosses that...it is not there anymore...

Figure 7-12 Typical gesture-drawing strategy

In situations when participants had to relate many objects or the picture of the results of moving different objects, some were observed using a pen-drawing strategy (Figure 7-13). A pen-drawing strategy is an event when participants imagine the behaviour of a graph by sketching. Nobody began imagining using the pen-drawing strategy. What usually happened is that they use other imagining strategies first and then at some point, they would dismiss looking at the screen on work on the Tablet PC. The participants were observed using this strategy to draw some lines. In the examples presented below, one participant (P1) drew lines after a series of gaze-drawing and mouse-drawing strategies. Also, P12 did the same in another task.



Static Chord

(00:06:21:12)

(P1 sketched some chords and indicated where the midpoints will lie)

P1: I am trying to imagine what happens as the lines tend to infinity...

Dynamic Tangent

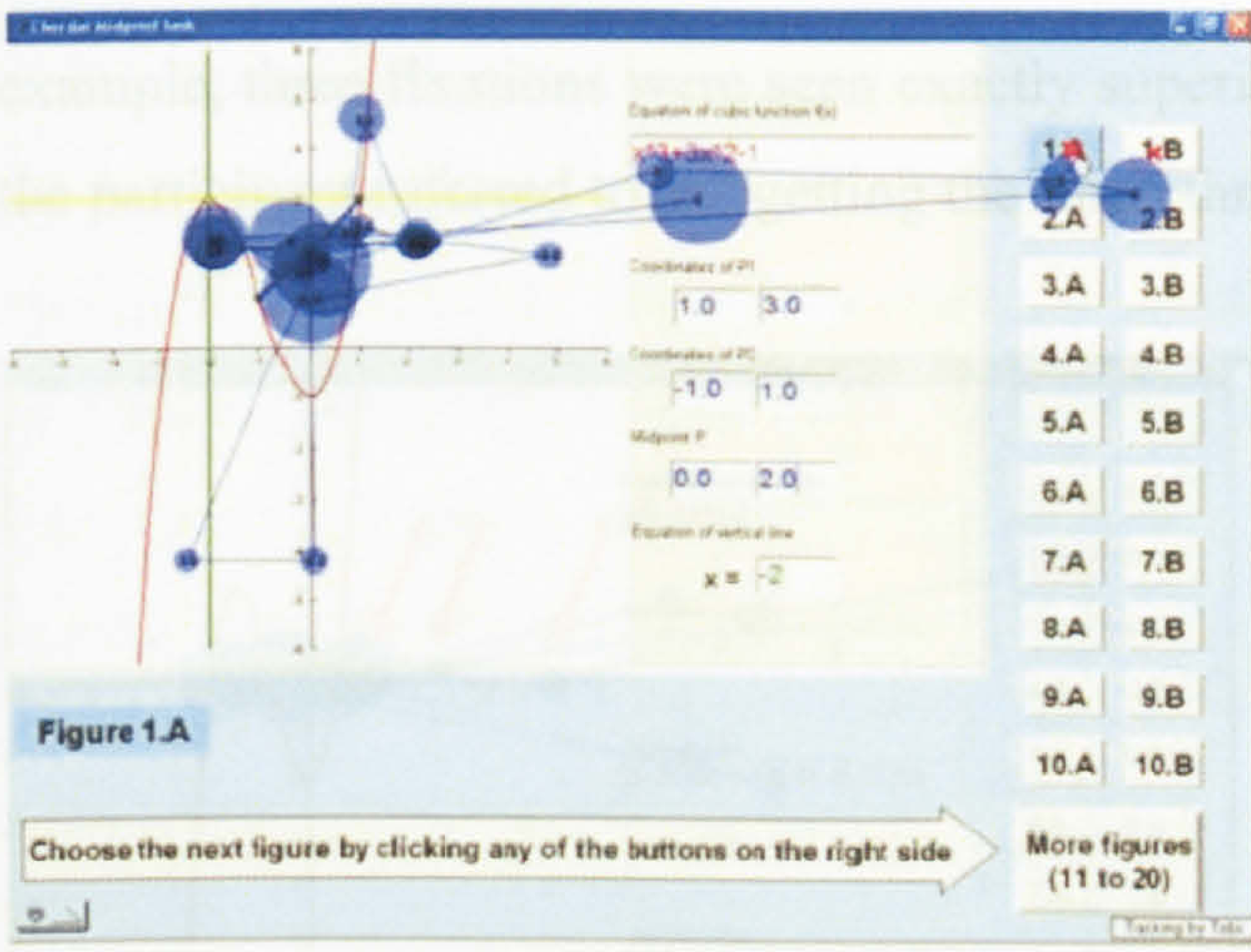
(00:06:28:22)

(Before P12 actually produced the 'marks' on the Tablet PC, the stylus pointer was observed moving upwards and downwards)

P12: There must be a line between there and there...

Figure 7-13 A pen-drawing strategy

The mental-drawing is characterised as an event when participants construct a mental representation and verbalises them. In the example given (Figure 7-14), it was difficult to tell whether the participant (P18) was doing a gaze-drawing strategy because the participant quickly looked at another figure. There were also some situations where there were indications when some participants can manipulate the graph in their head. In the other example below (Figure 7-15), the participant (P1) was able to tell what would happen to an object when it moves towards another object. There was neither any indication of indication of gaze-drawing nor a mouse-drawing that depicted what was being said. There is not enough evidence to compare this strategy with the other strategies.



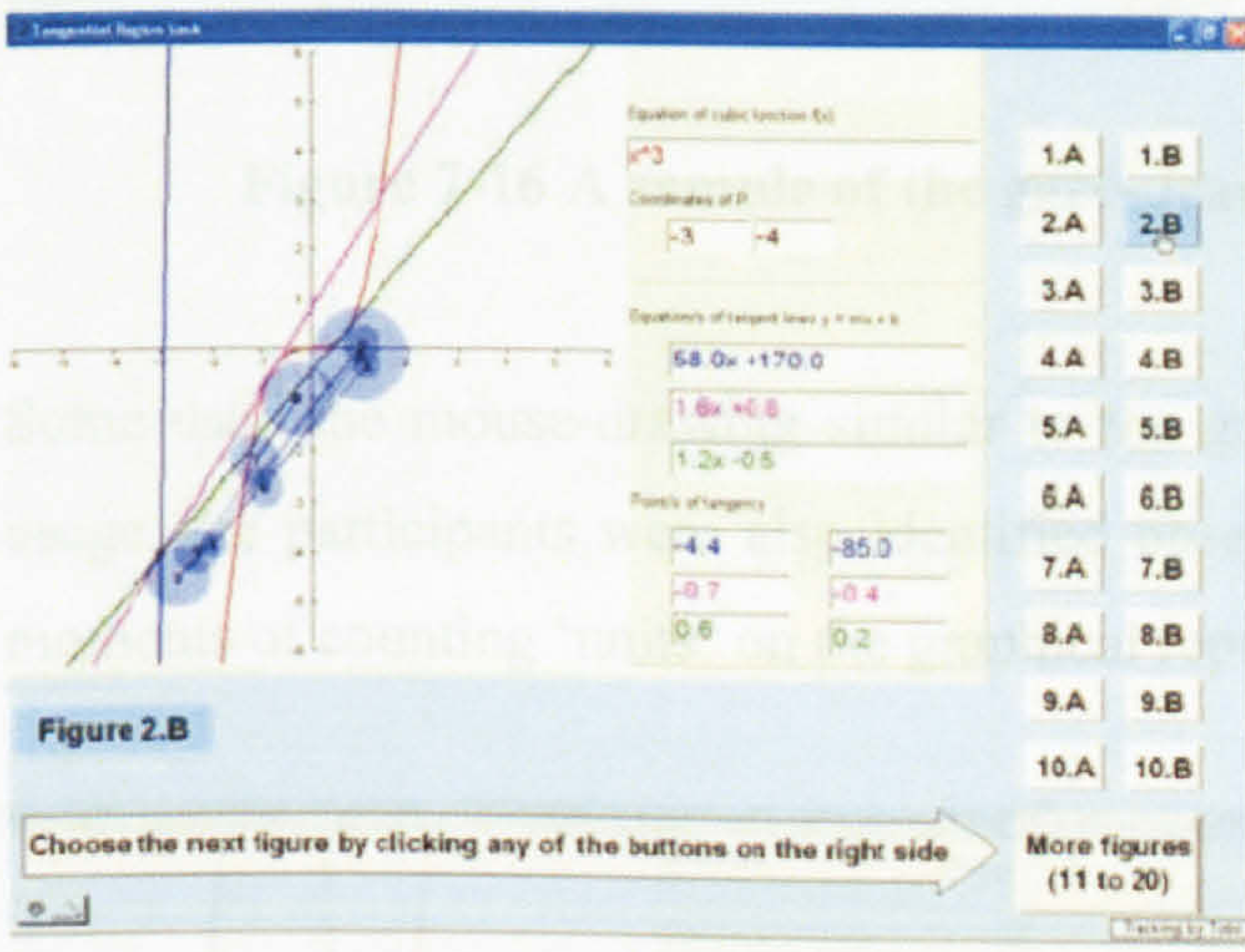
Static Chord

(00:01:35:02)

(P18 looked at one figure then quickly clicked on the next figure)

P18: I'm just looking at between these two… I'm trying to visualise in my head where all the midpoints could be…

Figure 7-14 A mental-drawing strategy in the Chord task



Static Tangent

(00:08:22:04)

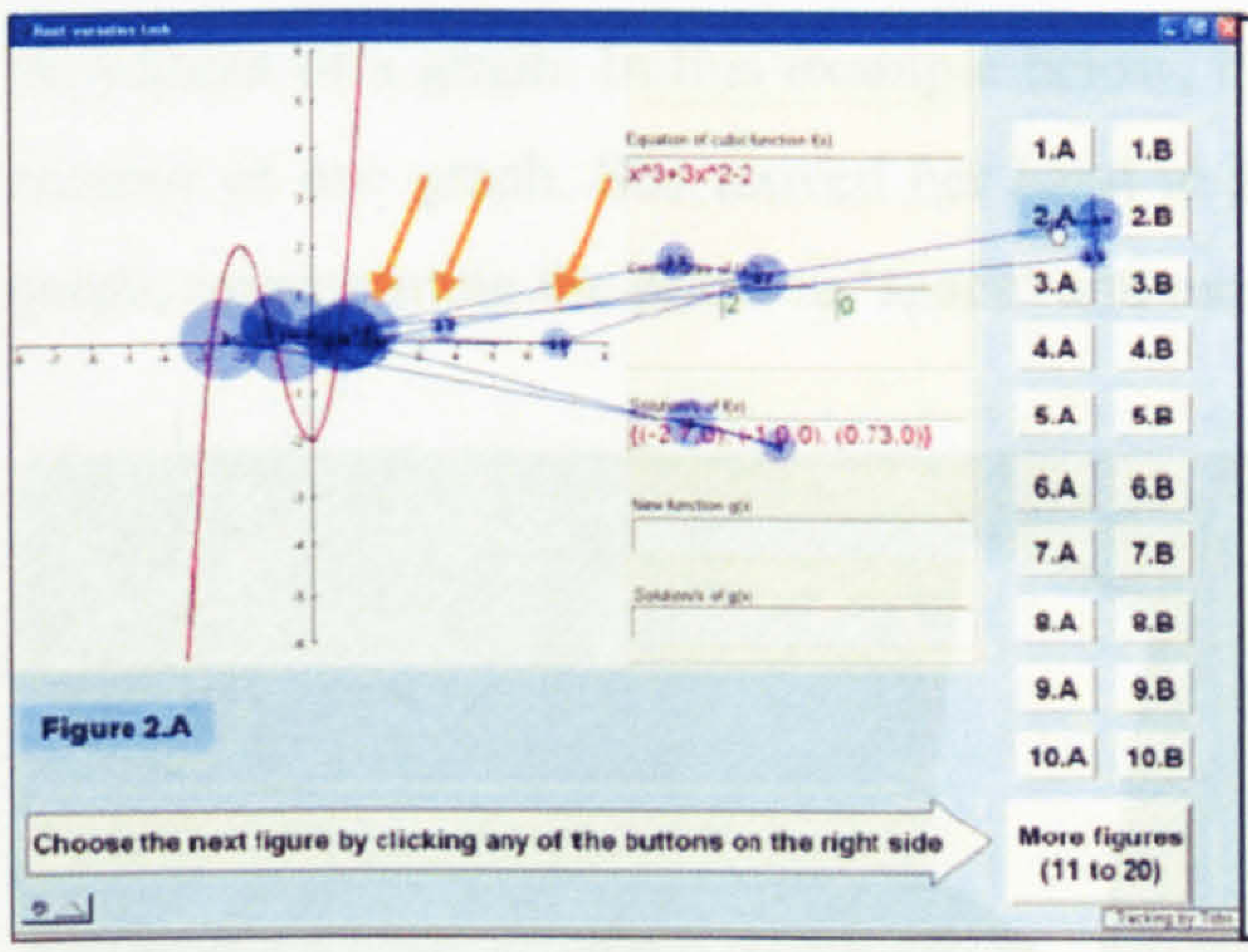
(P1 looked at the graph)

P1: When you are on that side of the graph you got three… as you move towards the graph it would have no effect on the green line… the pink one would no longer exist… and the blue would no longer exist

Figure 7-15 A mental-drawing strategy in the Tangent task

Imagining strategies in the simple task, the Root task

There were only a few imagining strategies that appeared in the Root task. The imagining strategies were applied differently. They used the imagining strategies to picture a perceived 'new location' of a graphical representation that appeared on the screen relative to its current location. The participants using this strategy in the Root task were found looking at something that is not there. They used this strategy in predicting where a graphical point will appear. A typical example is shown in Figure 7-16 below. In the example, three fixations were seen exactly superimposed on the parts of the screen which the participant referred to as "getting the same three".



Static

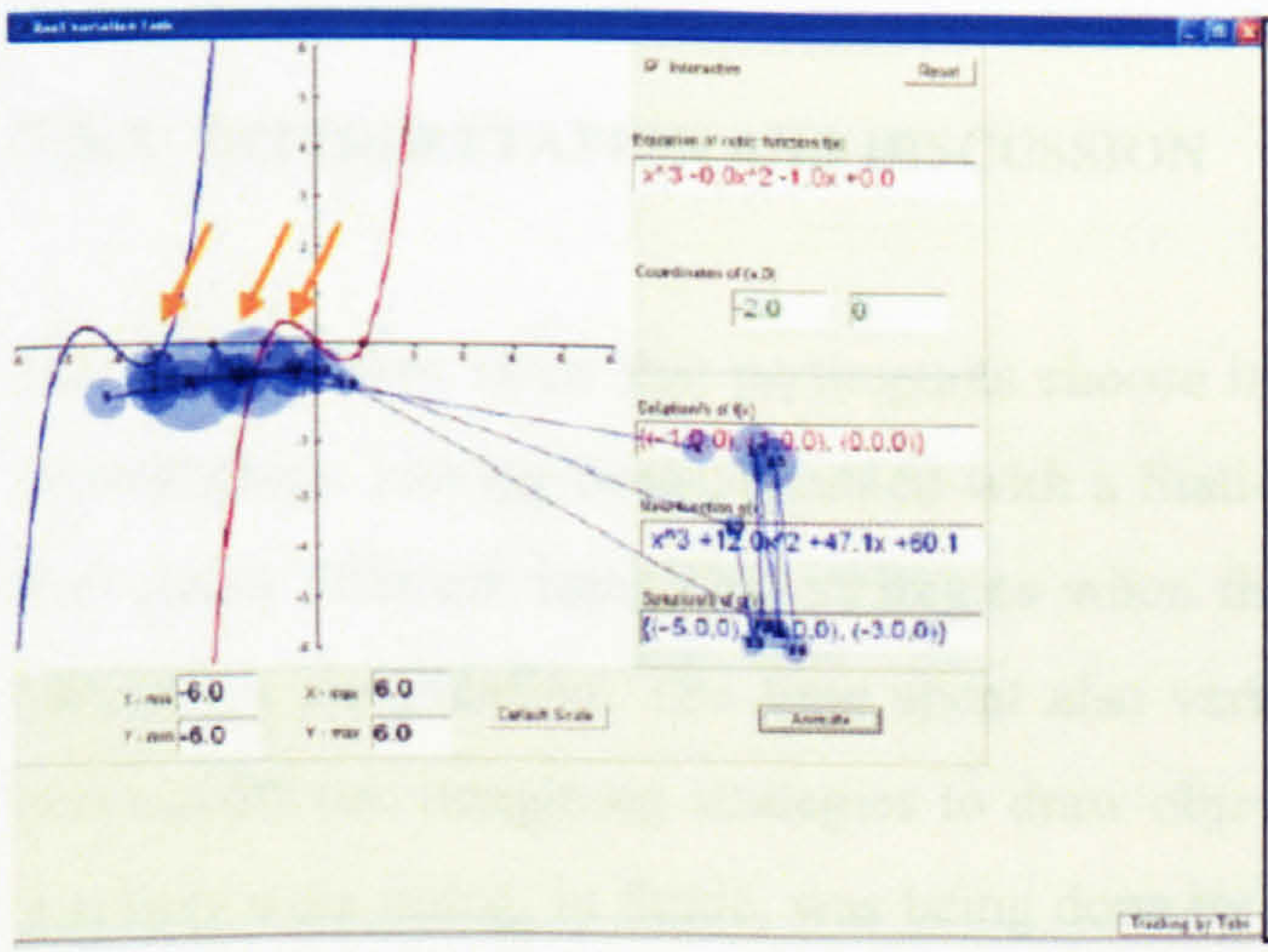
(00:01:29:19)

(P1 looked at the graph and looked at what is not there on the right side of the horizontal axis)

P1: "There are three solutions... there it crosses the x axis... once it has been rotated you get the same three... the three solutions are the same distance away from the point of rotation"

Figure 7-16 A sample of the gaze-drawing strategy in the Root task

Some used the mouse-drawing similar to the gaze-drawing above. Aside from the said usage, the participants were also identified observed using the mouse-drawing to depict moments of counting 'units' on the graphical representations (Figure 7-17).



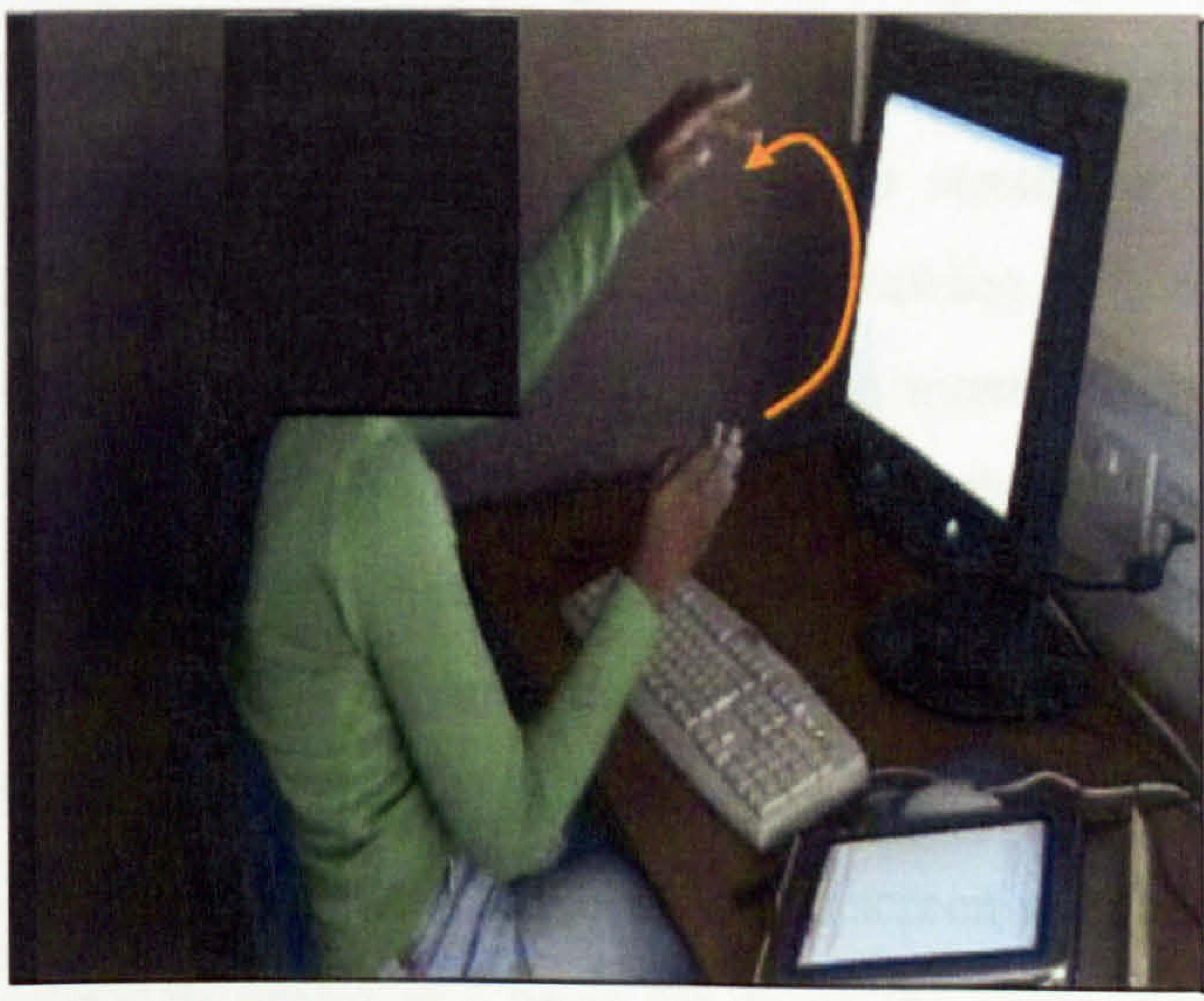
(00:10:14:11)

P6: Taken 5 from that one... one two three four... oh no...that's zero three... one two three...

(He used the mouse to count the number of distance away from a point)

Figure 7-17 A sample mouse-drawing strategy in the Root task

Some participants were observed using their hands to illustrate how the graph moved about the rotation point (Figure 7-18) and how a graphical point moved along the horizontal axis. The data show that the gesture-drawing was used to describe the movement of a graph. In this example below, P4 was observed placing her hands on the contour of one graph. She moved her hand in a circular manner as if rotating the entire graph, manipulating the graph in 'space' and moved the graph to its 'new location'.



Dynamic

(00:00:55:22)

(P4 moved her hands in a circular manner)

P4: It is going to go... If it rotates a hundred eighty degrees it is moving this way... moving this way... I think it is not going to change so much

Figure 7-18 A typical gesture-drawing strategy used in the Root task

7.3.3 Interpretation and Discussion

The results above show that participants choose imagining strategies differently between instantiations. Having been presented with a Static instantiation initially, the participants then chose different imagining strategies when they were presented with a Dynamic or Interactive instantiation. The time spent also varied. In Static, the evidence shows that participants use imagining strategies to draw objects. It seemed that the kind of drawing that they were doing, in Static, was being done by the animation features in Dynamic and the dragging feature in Interactive. For example, in the Chord task in Static, participants were observed using their eyes to depict the movement of a fixed graphical point. The participants do not need to do this in Dynamic or in Interactive because they can animate or manipulate objects. This suggests that the animation feature in Dynamic and the draggable objects feature in Interactive take the place of some of the imagining strategies used in the instantiation. Learners have been found to experience difficulty in constructing multiple images to create a moving image mentally (Ardac and Akaygun, 2005). Thus, visualisation difficulty may have been lessened when the instantiation is Interactive.

There seems to be a progression in using imagining strategies. Participants typically begin using either a gaze-drawing or mouse-drawing strategy. They attempt to stick with the two strategies first but when these strategies fail they shift to the other imagining strategies, such as gesture or pen-drawing. When a participant starts with an immediate inference about visual images using mental-drawing or without any indication of using the other imagining strategies, it seems that they have experienced this drawing situation before and may have gained expertise in manipulating visuals in their head. The evidence shows that participants use their eyes or the mouse to draw objects one by one; and then, they try to make an inference after several gaze or mouse drawing attempts. When they want to compare the objects on the screen with another object, they seem to use gesture to create an object as a basis for doing the comparison. It seems that when participants used a pen-drawing strategy, they were comparing or trying to make sense of many objects. This is difficult to do with gesture, eyes or mouse. This situation arose when participants decided to stop looking at the screen and work on the Tablet PC for some time. On the Tablet PC, they sketched and concretised what they had been imagining using their eyes, their hands, or the mouse.

The sequence of using imagining strategy from eye or mouse to gesture or pen may be related to increasing cognitive offloading. It could be speculated that the use of different imagining strategies to move objects either visually, mentally, or by external action is an indication where difficulty with visualisation is happening because learners have been found to experience difficulty in constructing images mentally and that certain interactive features can reduce this difficulty (Ardac and Akaygun, 2005). Learners have been found to experience difficulty in visualising movements of graphical representations (Lowe, 2003). So, in tackling computer-based representations, before learners use gesture and pen to address their difficulty in visualising movement of graphical representations, they have the option to use the mouse or their mind. In the videos of gazes and screen capture, it was possible to tell that this was the case. Godin-Meadow et al. (2001) suggest that gesture reduces cognitive load and also Cox (1999) suggest that the construction of representation in paper is a kind of offloading mental animation difficulty. The evidence presented above shows that some of the gesture was used to create a visual image of a graphical representation and pen-drawing used to capture movement, of graphs viewed on the screen, on paper.

Nevertheless it could be argued that some gestures may have been constructed by the participants as a form of communicative device to explain what they mean to the researcher. However, when participants made their gesture-drawing, the data record appears to suggest that they were involved with the task and using the gesture to explain something to themselves rather than explaining themselves to the researcher. On occasions participants did not show any use of strategies like gaze-drawing, mouse-drawing, or gesture and pen-drawing: one interpretation of this is that the participants have experienced the kind of representations before and they could probably easily create a mental picture in their head. This would be an indication of 'expertise'.

7.4 ATTENTION PAID TO REPRESENTATIONS

In section 7.2, the use of representation-specific strategies was seen to vary between instantiation in the simple task. The qualitative analyses in section 7.2 also show that participants' gazes to numeric, algebraic and graphic representations vary between instantiations. This section further addresses these findings and also provides more evidence that can support the hypothesis that attention paid with each standard external representation varies between instantiations. It seems that the different kinds of interactions in different instantiations encouraged participants to focus their attention on different aspects of the given representations. This section shows participants paid different attention to graphs and numbers. The comparisons are on the two prominent strategies that are mostly used: graph-wise and graphic-numeric in the two given tasks that apparently elicited only graphic-related strategies.

7.4.1 MAIN FINDING

In the previous chapter, it was noted that attention to external representations varies between instantiations within the complex tasks and that participants tend to pay much attention to graphical representations. Further qualitative analysis, as evidenced by scanpaths (see below), shows that the majority of attention is given to the graph but varying the instantiation given, can direct attention to important aspects of a graph, and can also direct attention to other representations.

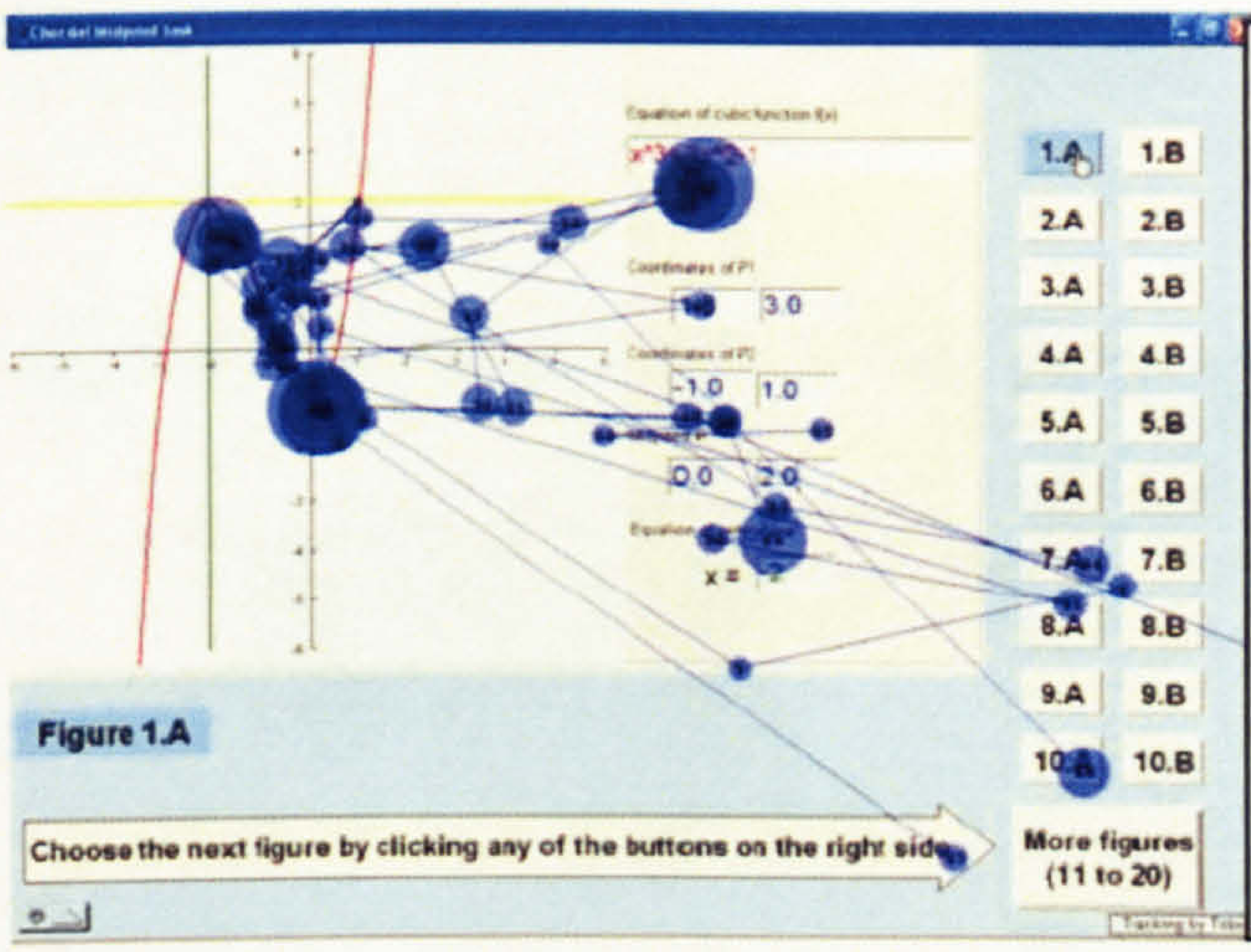
In Static, participants tended to focus their attention on representations that they perceived would change and the corresponding areas where a perceived movement of that change is going to happen. For example, if they expect a point to move towards the right, their gazes will occupy that right-side area. In Dynamic, participants' attention focused on entry and output. Whilst in Interactive, the focus of attention is on what is being manipulated (e.g. a point being dragged about).

7.4.2 EVIDENCE

First, examples of the analysis in illustrating gazes to representations in Static is presented; and then followed by the analysis in Dynamic instantiation. The evidence gives indication on how attention paid to representations relates to how participants tackle graph representations and strategy use.

Attention paid to representations in Static

In Static, the participants paid attention to the overall appearance of the graph. They tended to describe the graph in terms of how the graph appears as in the example given in Figure 7-19. As the participant looked at the graph, P18 used the graph-wise strategy by giving his perception of how the graph would appear.



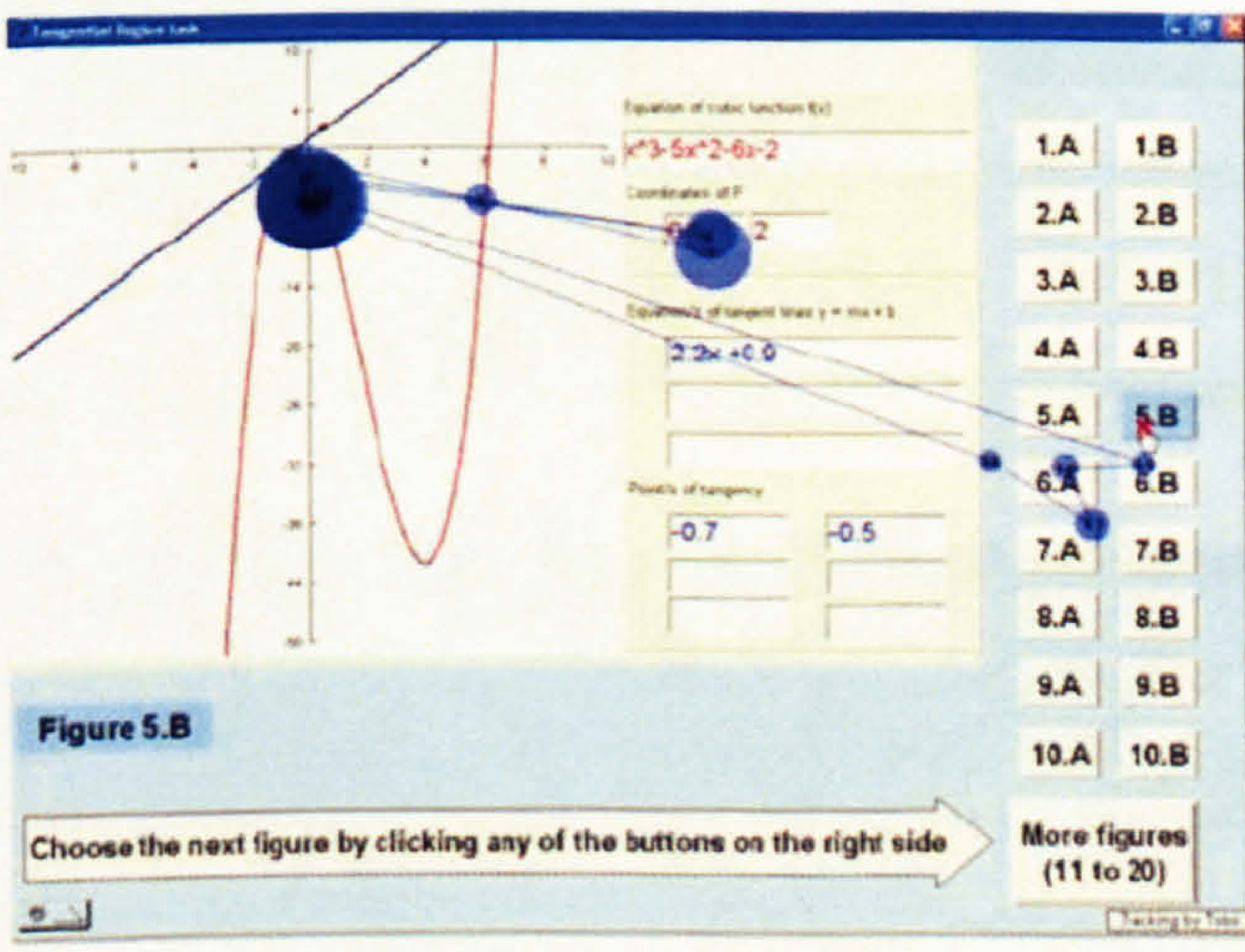
Static

00:10:16:07

P18: Two lines bound to this shaded area... I think it would be parallel... if this is symmetrical function then it would be parallel... the lines... the minimum and the maximum would be stretch in that way...

Figure 7-19 A description of a graph based on the attention given to its appearance

At times when participants pay attention a particular element of the graph, in Static, the participants used the graph-wise strategy to describe a perceived movement of a graphical element. In the particular example (Figure 7-20), the participant described the movement of a graphical point either left to right or going up to going down.



Static

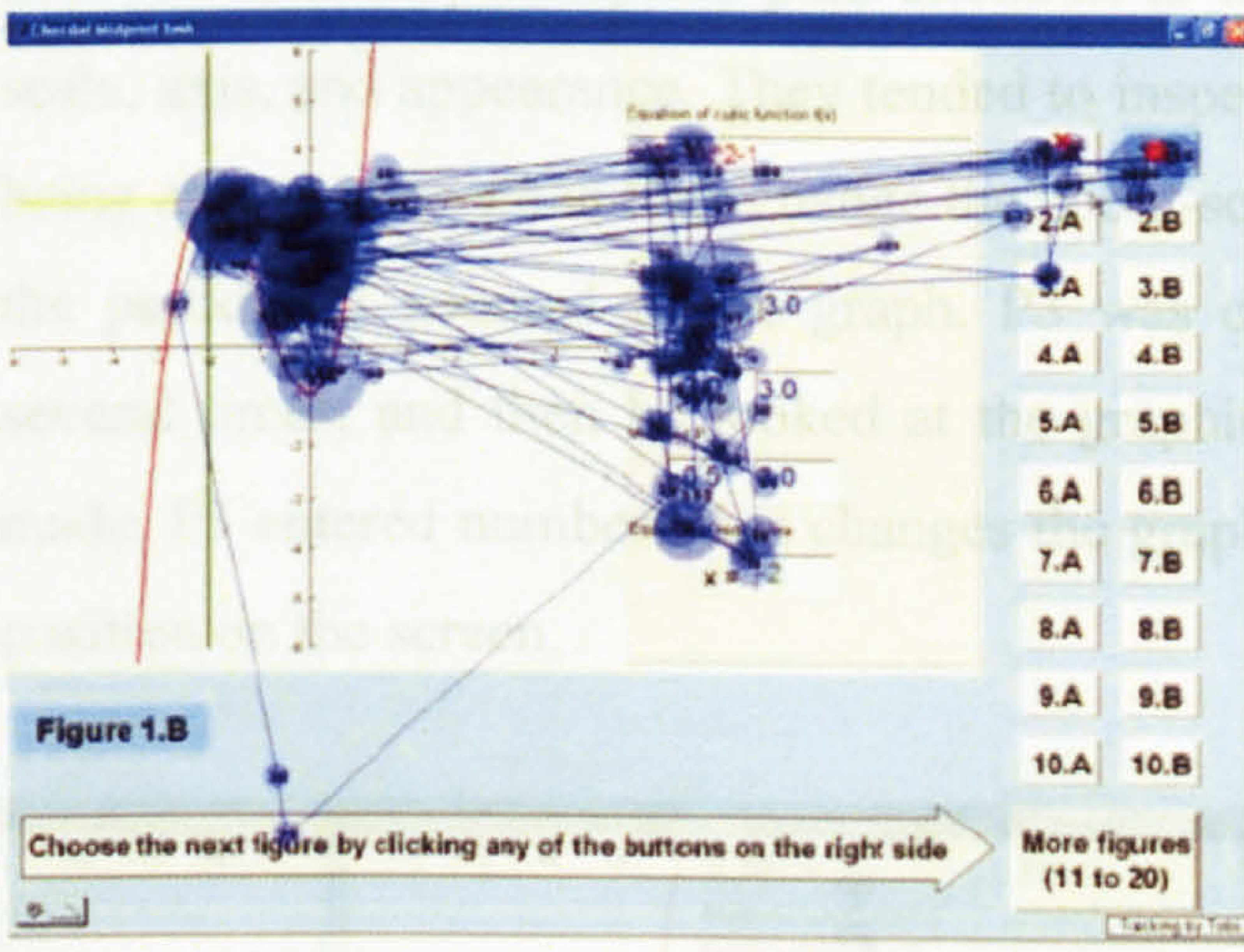
(00:11:29:06)

P1: So that point moved only in the x direction..."

(P1 described the movement of the point)

Figure 7-20 A description of movement of a graphical representation

The participants also paid attention to the graph and numbers when they use a graphic-numeric strategy. The participants, in Static, matched the numbers with the corresponding graphical equivalent (Figure 7-21). In the example, when the participant looked at an ordered number pair (-2, 3), she also looked at the corresponding graphical point. P1 did this for each of the numerical representations and tried to come up with an inference about the two representations.



Static

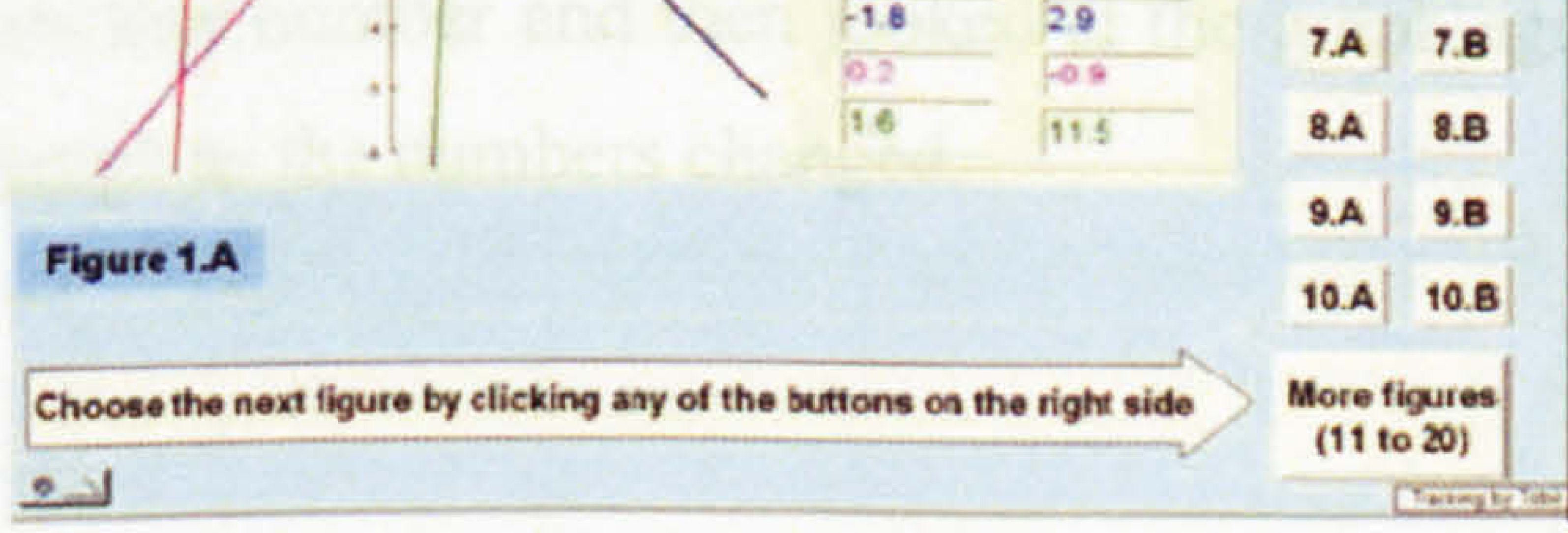
(00:00:50:08)

P1: The point one is there... the midpoint is there... the yellow line is not doing much I think...

(P1 looked at the graph and the numbers)

Figure 7-21 Attention paid to the graph and number

The participants also used the graphic-numeric in Static, by inspecting the number in the ordered pair and relating it to a perceived location on the graph when the number was changed. This number corresponds to the number that can be entered in Dynamic and also the number that changes, in Interactive, when a graphical point is dragged about the screen. In the example below (Figure 7-22), P13 was observed making sense of how the graph would change if the number (1, 0) changes.



Static

00:08:24:08

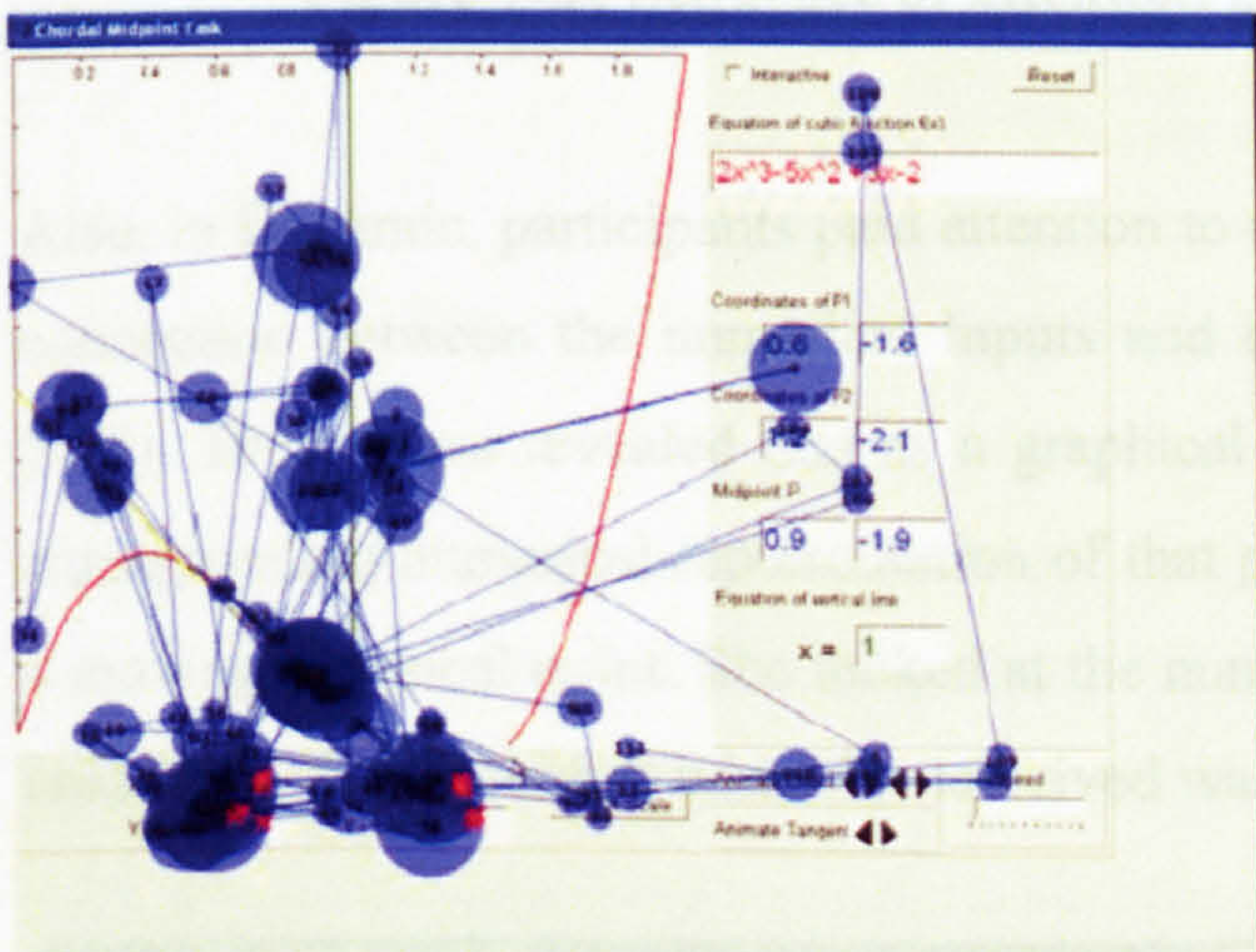
P13: Zero to minus one... it is one...no minus to one it is one...

(P13 looked at different figures and looked at the location of the tangents in relation to minus one, zero, and one)

Figure 7-22 Attention paid to two corresponding representations

Attention paid to representations in Dynamic

In Dynamic, the participants paid attention to the details of the graph in relation to its scale, axis, and appearance. They tended to inspect the details of the graph as the graph is being animated or when they change the zoom-scale. In the example below (Figure 7-23), the participant zoomed in the graph. P3 was observed changing the numerical inputs several times, and then he looked at the graphical counterpart before an inference was made. P3 entered numbers that changes the graph as it gets closer and closer to a certain position on the screen.



Dynamic

00:11:49:07

P3: Can I zoom it... hmmm... that midpoint is somewhere... where I am not expecting it to be...

(P3 described the location of the midpoint)

Figure 7-23 Attention paid to change in scale

Sometimes participants do not play the animation; rather, they input numbers successively. The participants pay attention to the graph after a numerical entry. The description of a graphical point was given every after numerical input. In the example below (Figure 7-24), P4 entered a number then looked at the graph, and then she entered another number and then looked at the graph again. She described what happened to the graph as the numbers changed.

Dynamic

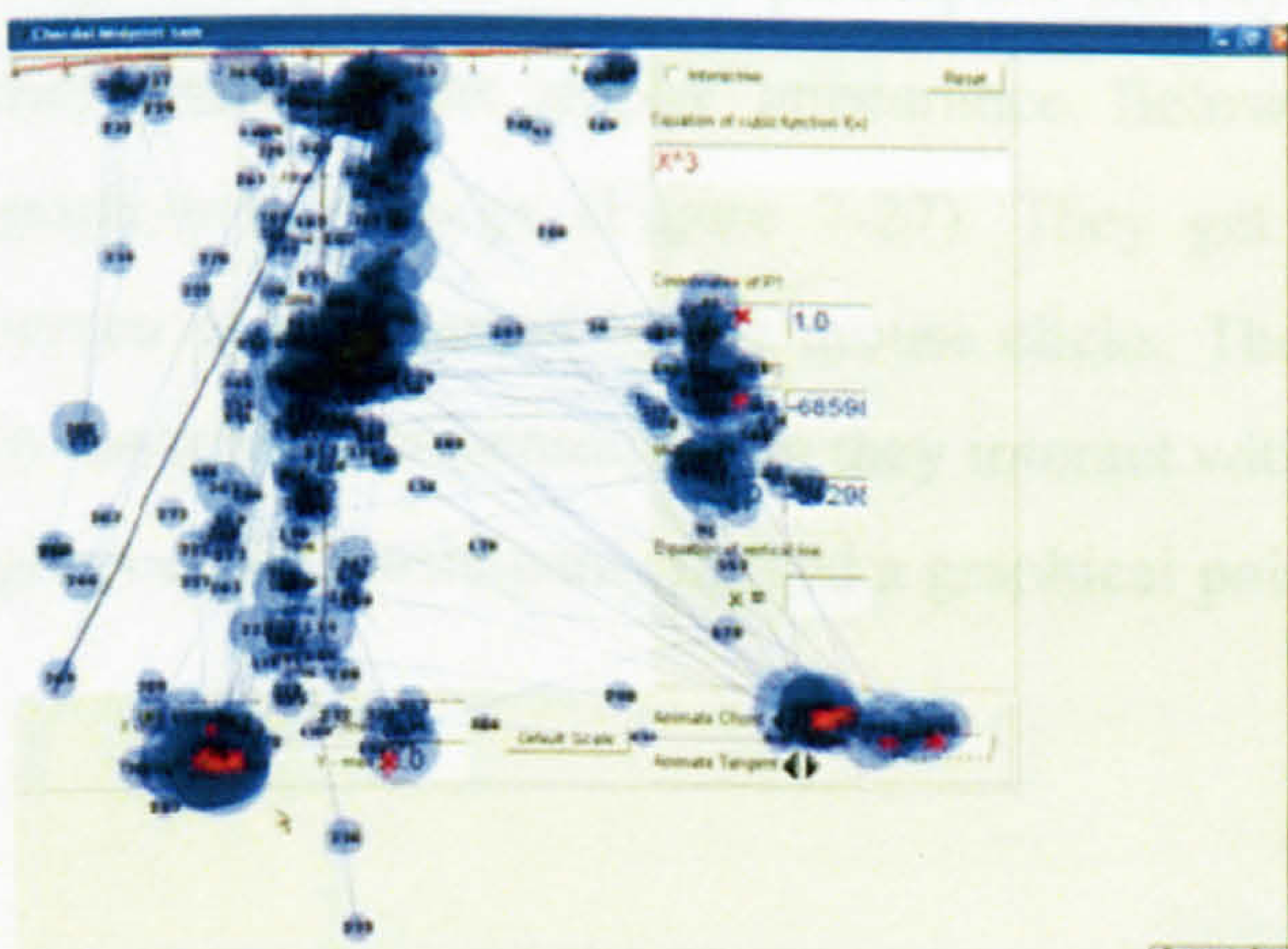
(00:02:58:17)

P4: zero one... alright so I'm guessing that for minus one it's the same thing... so that is going down...Err... I'm going to try something down here"

(P4 entered zero one. She described the position of the tangent)

Figure 7-24 Sequence of attention given to graphs and numbers

Also, in Dynamic, participants paid attention to representations when they were making a connection between the numerical inputs and the movement that graph makes (Figure 7-25). Their gazes revealed that as a graphical point is moving, they also gazed at the corresponding numerical representation of that point. P4's eye-marks observed following a moving graphical point. She looked at the numbers once in a while whilst watching the animation. She described what she perceived was happening.



Dynamic

(00:09:37:00)

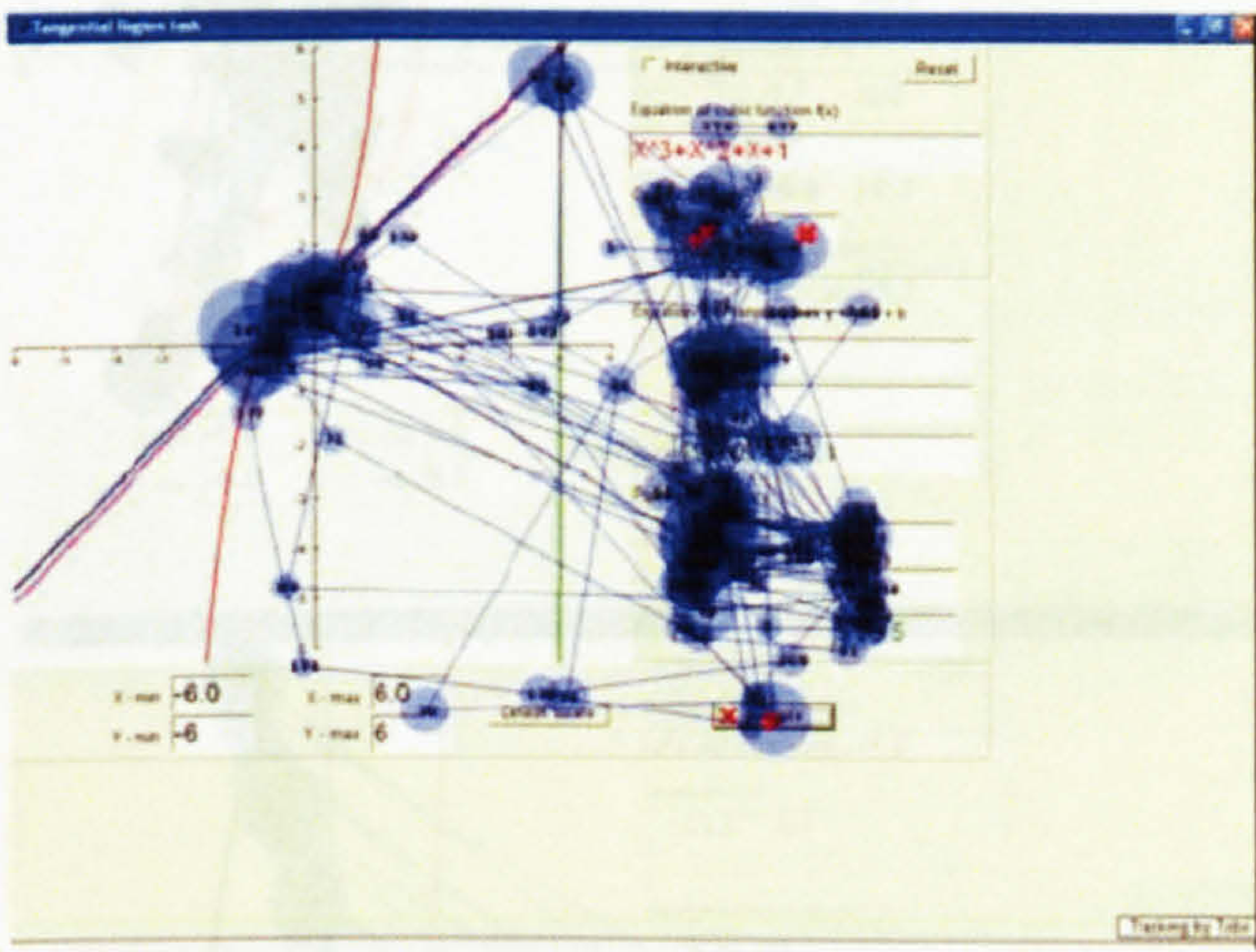
P4: Ah! Ok...using minus one to one... It goes down that way... So it crosses the line... But... it cuts one zero... So that's down right...and it still keeps going...

(P4 animated the graph in different directions. She looked at the graph and numbers back and forth as she watched the graph. She changed the zoom scale)

Figure 7-25 Attention given to a graph and an automatically changing display

There were events when participants in Dynamic recorded the numbed entered using the software, on the Tablet PC. The participant compared the numbers they recorded on the

Tablet PC with what was entered on the screen, similar to the example given below (Figure 7-26).



Dynamic

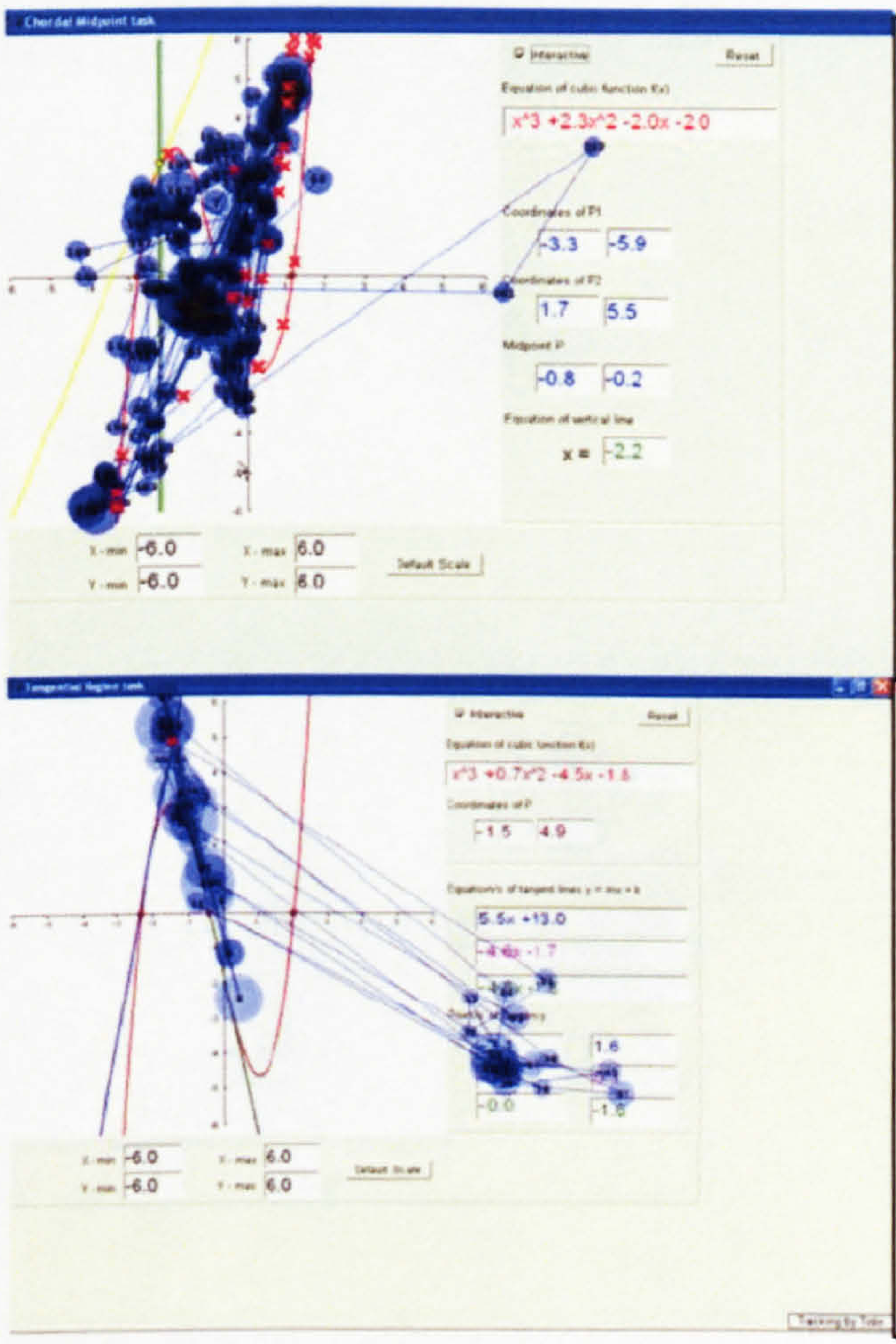
P18: I'm gonna go back and look at some of the other points that I have put down here already... When you look at the points of tangency... see if I can work out the line...

(P18 looked at the Tablet PC. Then, he entered different coordinates)

Figure 7-26 Attention given to recorded representations

Attention paid to representations in Interactive

In Interactive, the participants inspected the graph by manipulating points that they can drag. It was observed that participant heavily paid attention on the graph alone. Then, they described the graphs' appearance. Below typical examples when participants use a graph-wise strategy (Figure 7-27). They get too involved dragging objects about the screen as represented by the mouse clicks. They tended to ignore the changes happening in the other representations as they interact with the graph. The description of the graph is given as the participants moved a graphical point.



Interactive

00:11:07:15

P12: The top boundary is kind of determined by... I wonder if I can get... Oh... it is probably more complicated... I bet you can get all of the points...I suppose it is difficult...

(P12 looked at the graph and draggerd the points many times)

Interactive

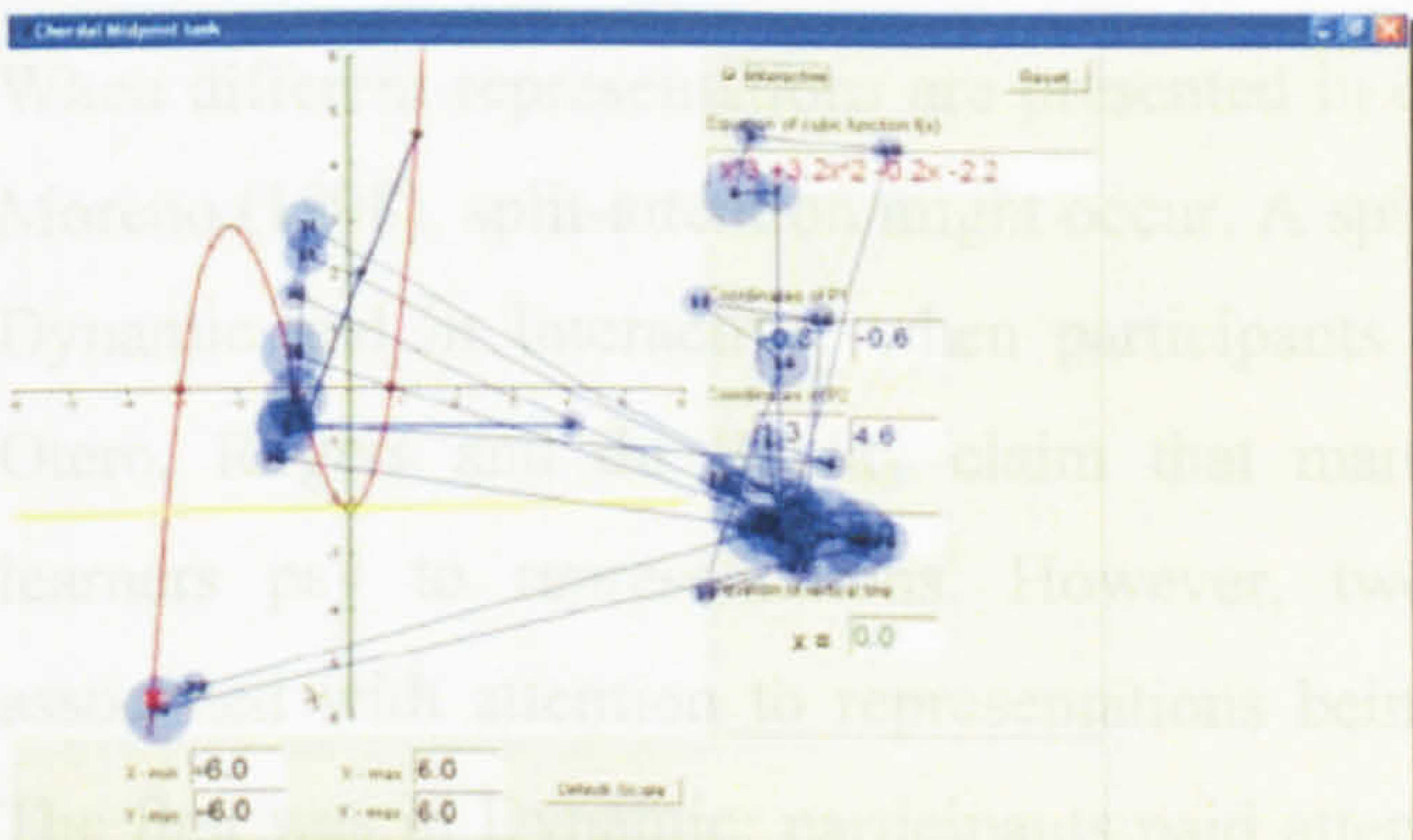
(00:07:19:10)

P17: I think anywhere outside... would give you three... I am assuming...

(P17 looked at the graph as he dragged the point. Then he sketched on the Tablet PC)

Figure 7-27 Attention on graphs whilst ignoring other representations

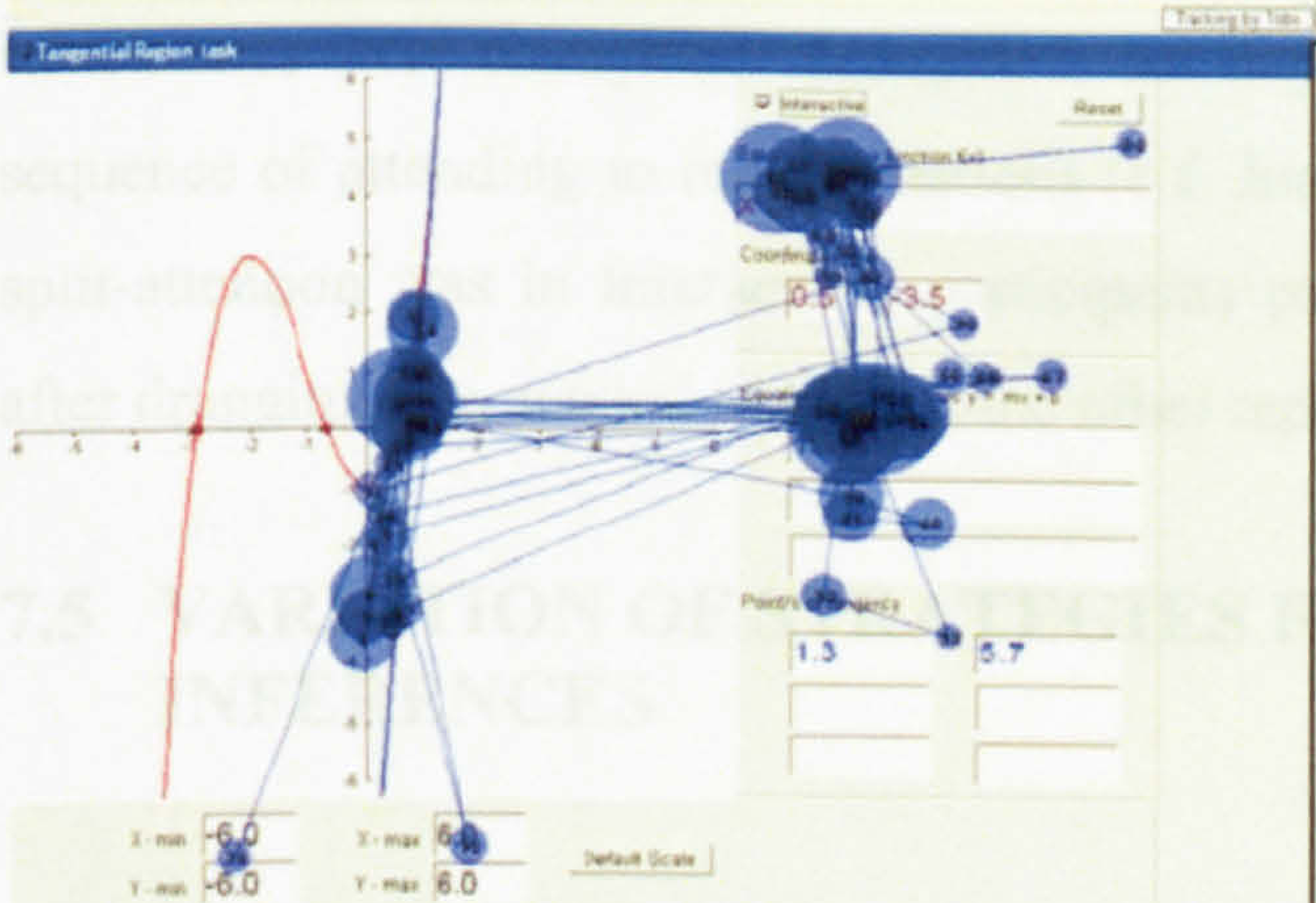
When participants use a graphic-numeric strategy, in Interactive, they tended to focus on the movement that a graphical point first and then they looked at the numbers as they drag the graphical point. The comparison between the numbers and the graphs were made as they move the graphical point. Similar to Figure 7-28 below, P6 shifted his attention to the numbers from the graph. P6 initially looked at a point on the graph, and then dragged it around. Then, his gaze concentrated on the numbers ignoring what was happening on the graph as indicated by the eye-marks. The same kind of situation was observed with P15 (below of Figure 7-28). P15 was observed tracing a shape by dragging a point vertically. Whilst doing it, she tried to keep the number constant as she moved the point. P15 failed in establishing a pattern based on the actions she did.



Interactive
(00:04:13:21)

P6: I'm looking at the coordinates of the points... Boundary can be a curve... as the function goes vertical it follows it... I should watch the numbers... I don't see anything from this...

(P6 looked at the numbers then the graph then the number again.



Interactive
00:03:20:10

P15: I can't work out what the tangent is when it switches from three to one...

(P15 tried to relate the graph with the coordinate of P as shown in her gazes)

Figure 7-28 Attention given to a graph and numbers one at a time

7.4.3 Interpretation and discussion

In section 6.2, the quantitative evidence shows that participants used graphic-related strategies prominently in completing the complex tasks. The evidence above shows that although participants did not use algebraic-related strategies or numeric-related strategies, their gazes, in using graphic-related strategies, vary between instantiations. Other empirical studies have shown that learners prefer graphical representations over algebraic or numeric (e.g. Keller and Hirsch, 1998; Villarreal, 2000). It seems that in Static, participants typically paid attention to the graph first and described its overall appearance. This may mean that when MERs are shown to participants, participants start attending to representations that they prefer. Participants then tend to use the other representations (numbers or equations) to supplement what they are seeing on the graph.

When different representations are presented in different forms, according to Mayer and Moreno (1998), split-attention might occur. A split-attention effect was seen happening in Dynamic and in Interactive; when participants either animate or drag certain objects. Otero, Rogers and du Boulay claim that manipulability can influence the attention learners pay to representations. However, two different kinds of findings can be associated with attention to representations being affected by the type of instantiation. The first was in Dynamic: participants paid attention to input and then to the output and also to the moving part of the graph and gazed at the other representations (depending on which ones were changing). It seemed that depending on the interactivity, there was a sequence of attending to representations (c.f. Jones and Scaife, 2000). The other kind of split-attention was in Interactive: participants paid attention to the graph first and then after dragging they tended to look at the other representations that had changed.

7.5 VARIATION OF STRATEGIES FOR EXPRESSING INFERENCES

The hypothesis considered in this section is that expression of inferences varies depending on the instantiation. The sequence of operation required due to the design of the computer-based MERs may also impact on the strategy that learners may apply. The three re-representing strategies identified were visual, textual and symbolic. These strategies are also related to external representations offered to the participants and since there are differences found in the use of representation-specific strategies as presented above (chapter 6 section 3), differences in the use of re-representing strategies can also therefore be expected to appear. Participants' recorded inferences are compared between instantiations. The main finding is presented first followed by some examples of inferences that provide evidence for the finding. The rest of the participants' written conjectures are given in Appendix E.

7.5.1 MAIN FINDING

Instantiation influences the way participants express their inferences. For example, in the Root task, if the instantiation is Static or Interactive, participants tend to make inferences that refer to graphs; however, in Dynamic, the inferences tend to be algebraic. In the other two tasks, if the instantiation is Dynamic or Interactive, participants tend to re-represent the 'movement' that a graphical representation makes; however, in Static, the inferences tend to represent the final perceived picture.

7.5.2 EVIDENCE

Screen capture of participants' recorded inferences is presented by task. The writing and sketching of the inferences in figures below are seen happening in real-time by watching the Tablet PC screen capture replay. The analysis of the writing video was also related to the action and gaze video.

Variation of inferences in the Root task between instantiations

In one of the tasks, the Root task, there is no evidence of visual re-representing strategies in Static. The constructed inferences were all in text. As reported earlier in section 6.2, participants' final conjectures were all correct in Static. However, their inferences were about descriptions of the graphical appearances. Five out of the six participants described either the graphical distances between two graphical points or stated that the points are reflections of each other. Two examples of these kinds of inferences are presented in Figure 7-29.

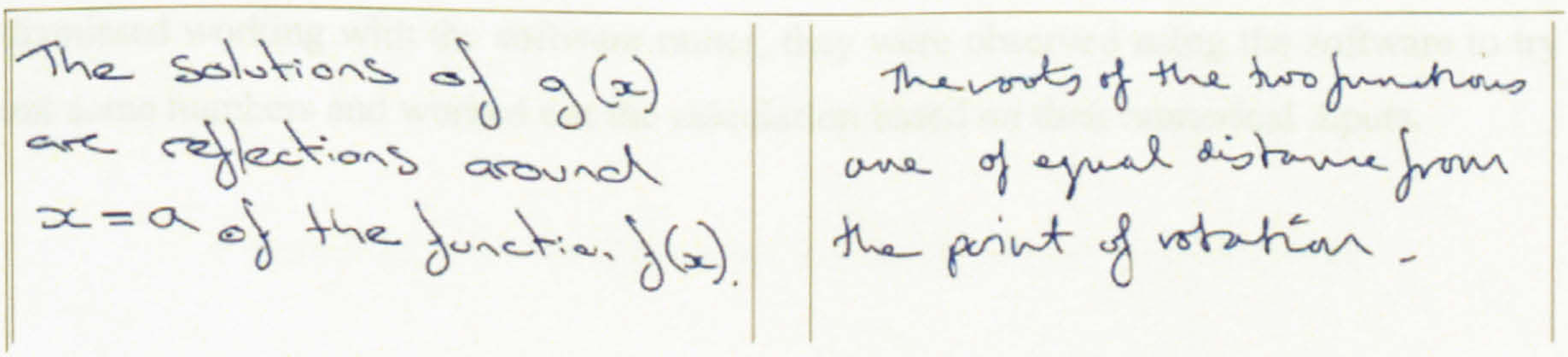


Figure 7-29 Written inferences for the Root task in Static

There was one participant atypical to the other five above. The atypical participant, in Static, arrived at an algebraic rule in the final conjecture (Figure 7-31). Unlike the participants who concentrated working with the software, this participant dismissed looking at the screen and worked closely on the Tablet PC.

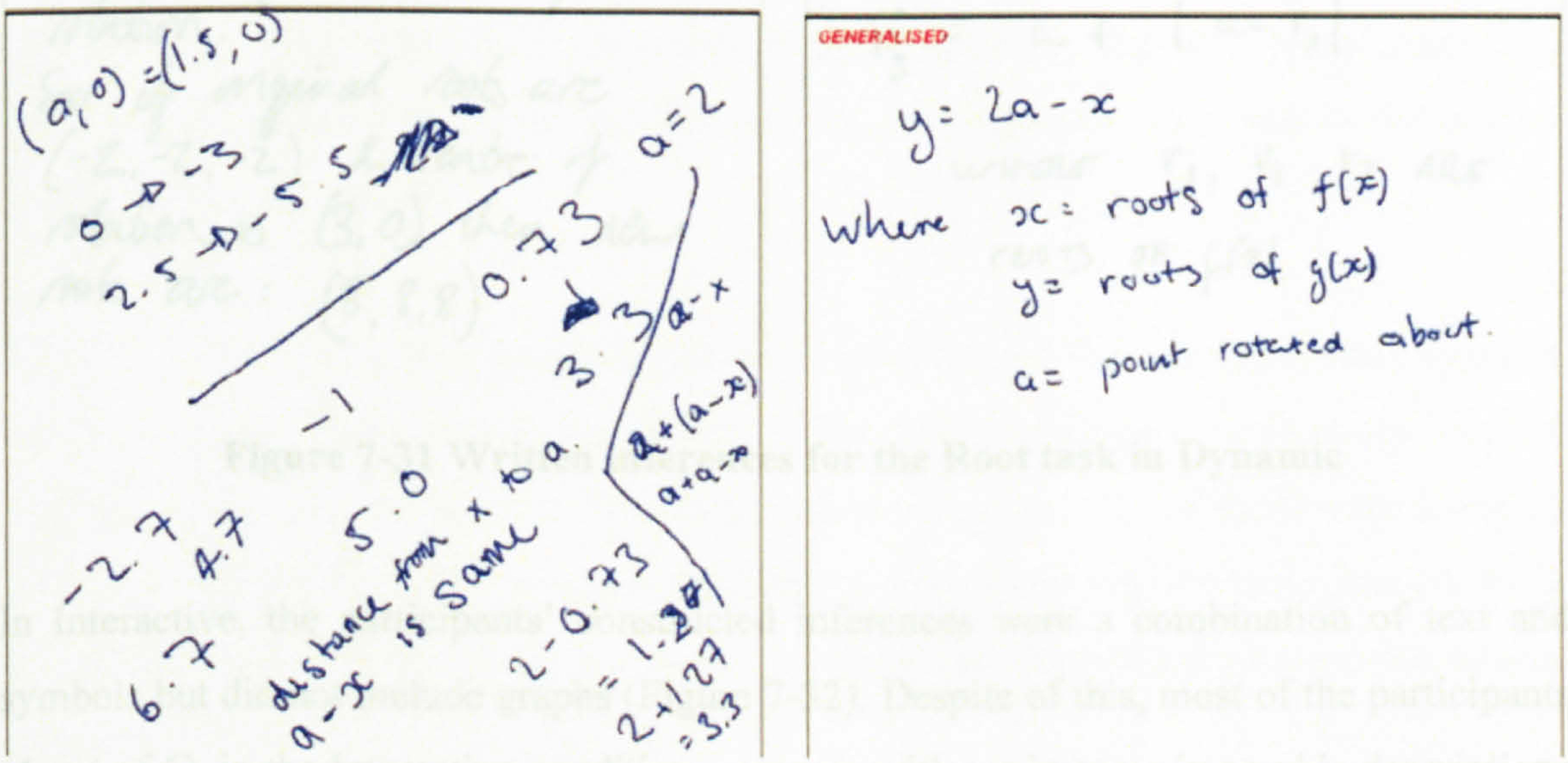


Figure 7-30 Written inferences of an atypical participant in the Root task in Static

In Dynamic, the participants' constructed representations were a mixture of symbols, text and graphs/arrow diagrams. Similar to the atypical participants above, all six participants in the Dynamic condition ended up with algebraic conjectures. Four out of six came up with a correct conjecture. Some expressed their conjectures in words whilst others in algebraic symbolisations (Figure 7-30). Nobody seemed to express their written inferences using visual strategy. The participants in Dynamic, however, did not totally dismissed working with the software rather, they were observed using the software to try out some numbers and worked out the calculation based on their numerical inputs.

roots are the inverse sign of original roots plus double the x value of the centre of rotation.
So if original roots are (-2, -2, -2) & centre of rotation is (3, 0) then new roots are: (8, 8, 8)

$r_1' = a + |a - r_1|$

$r_2' = a + |a - r_2|$

$r_3' = a + |a - r_3|$

WHERE r_1, r_2, r_3 ARE ROOTS OF $f(x)$.

Figure 7-31 Written inferences for the Root task in Dynamic

In Interactive, the participants' constructed inferences were a combination of text and symbols but did not include graphs (Figure 7-32). Despite of this, most of the participants (4 out of 6), in the Interactive condition, came up with conjectures in graphic descriptions whilst others (2 out of 6) came up with conjectures based on algebraic rule. The two participants who arrived with an algebraic conjecture were also considered as atypical to others in Interactive because they came up with an algebraic rule by working with the Tablet PC.

The point (a,0) defined a vertical symmetry axe.
For each root of f(x), g(x) has a root which is the symmetric point relative to axis vertical defined by (a,0).

when f(x) is rotated 180° about (a,0) the roots of the new function are (-x+2a, 0) where x is the value of the original root.

Figure 7-32 Written inferences for the Root task in Interactive

Variation of inferences in the Chord task between instantiations

For the chord task, the difference found between instantiations is in the use of point-wise in making inferences. Many participants in Static and Interactive related their inferences to certain discrete points of a graph but not in Dynamic. The details of the analysis are presented below. In the Chord task, no symbolic re-representing strategy was apparent. The textual and visual re-representing strategies appeared across instantiations. In the Chord task, the results show that there is a difference in the written inferences between instantiations. Although nobody answered the task correctly, the ways participants used the strategies differ in terms of whether they are relating their inferences to the discrete points of the graph or not.

For the Chord task, nobody answered the task correctly. All the inferences were based on text and graphic construction (e.g. Figure 7-33). The participants' conjectures were all found to relate their answers to discrete points of the graph. All of them were using the strategies to describe their inferences by drawing chords and relating their answers to the discrete points of the graph. In the figure, the particular discrete points were the minima and the maxima.

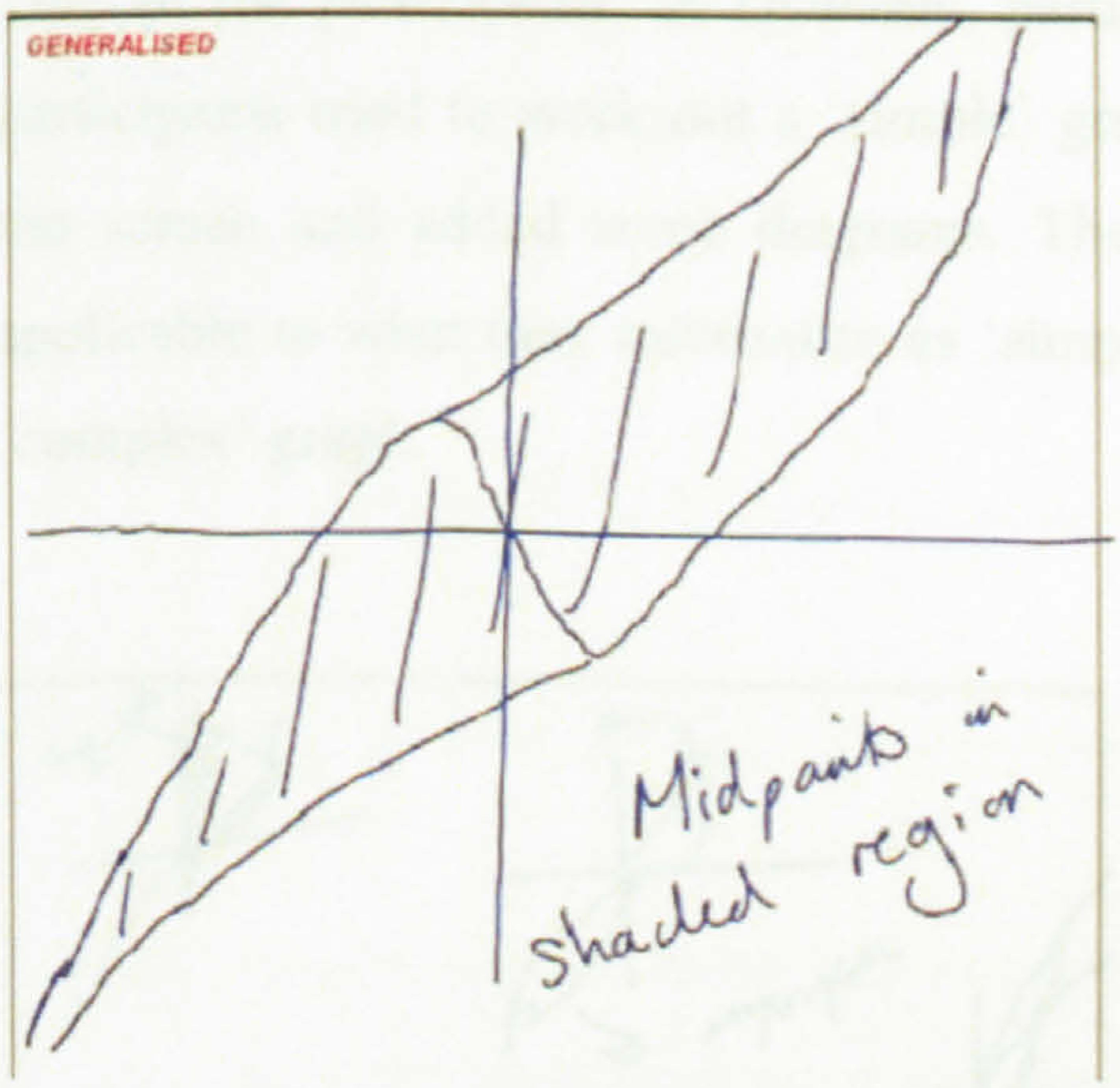


Figure 7-33 Written inferences for the Chord task in Static

In Dynamic, the written inferences show that nobody related their inferences to the discrete points of the graph. Instead, the participants sketched on the Tablet by tracing the path that a graphical point made on the screen. In the examples below, the participants' gazes fixated on a moving point and then drew the movement that a graph made whilst watching the animation. The arrow indicating the traced 'figure' as if participants'' sketched were based on 'freeze frames' of an animating graph (similar to that of the Figure 7-34).

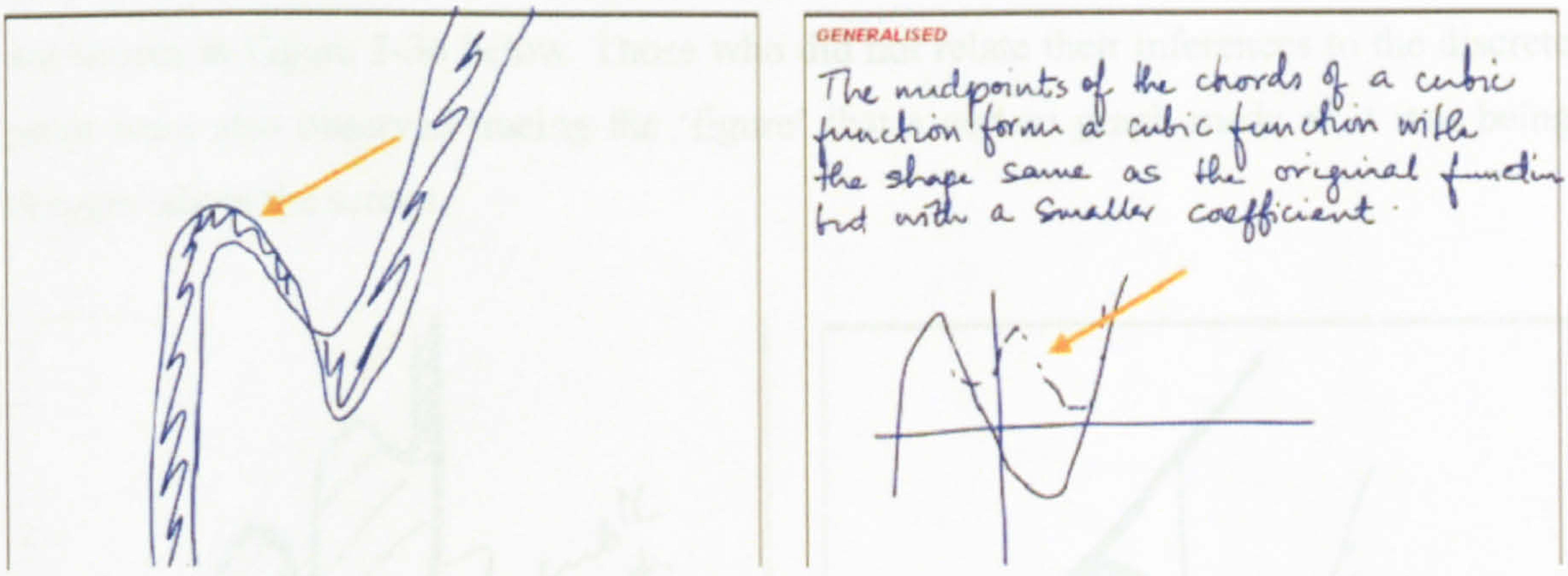


Figure 7-34 Written inferences for the Chord task in Dynamic

Two of the participants, in Dynamic, partially answered the task correctly. These two participants tried to work out a 'simple' graph (Figure 7-35). They copied what was on the screen and added some diagrams. They were able to identify the correct answer applicable to what they referred to as 'simple' graph but failed to relate this answer to a 'complex' graph.

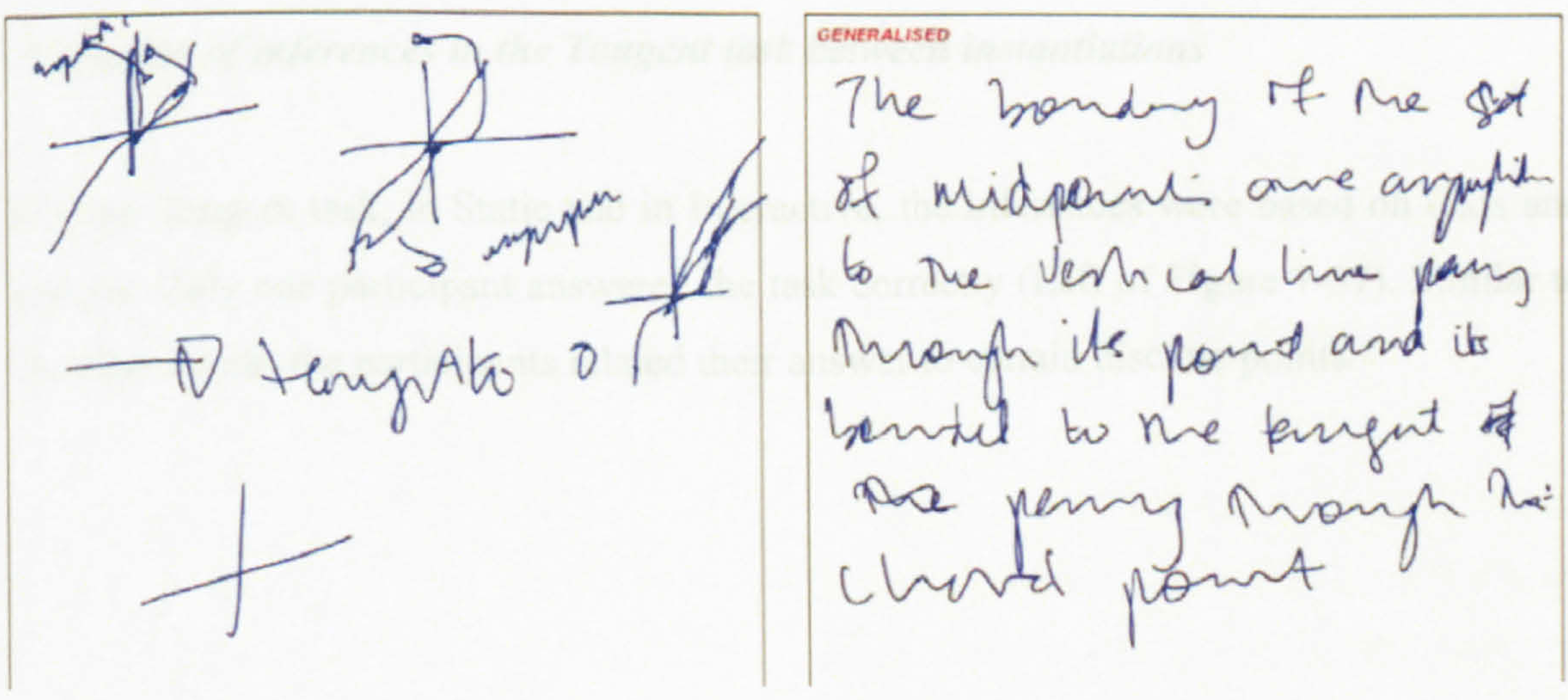


Figure 7-35 Written inferences for the Chord task in Dynamic

In Interactive, the answers were all incorrect. Four of the participants related their inferences to the discrete points of the graph, similar to the participants in Static; whereas two did not, similar to the participants in Dynamic. Examples of inferences in Interactive are shown in Figure 7-36 below. Those who did not relate their inferences to the discrete point were also observed tracing the 'figure' that a certain graph made as it was being dragged about the screen.

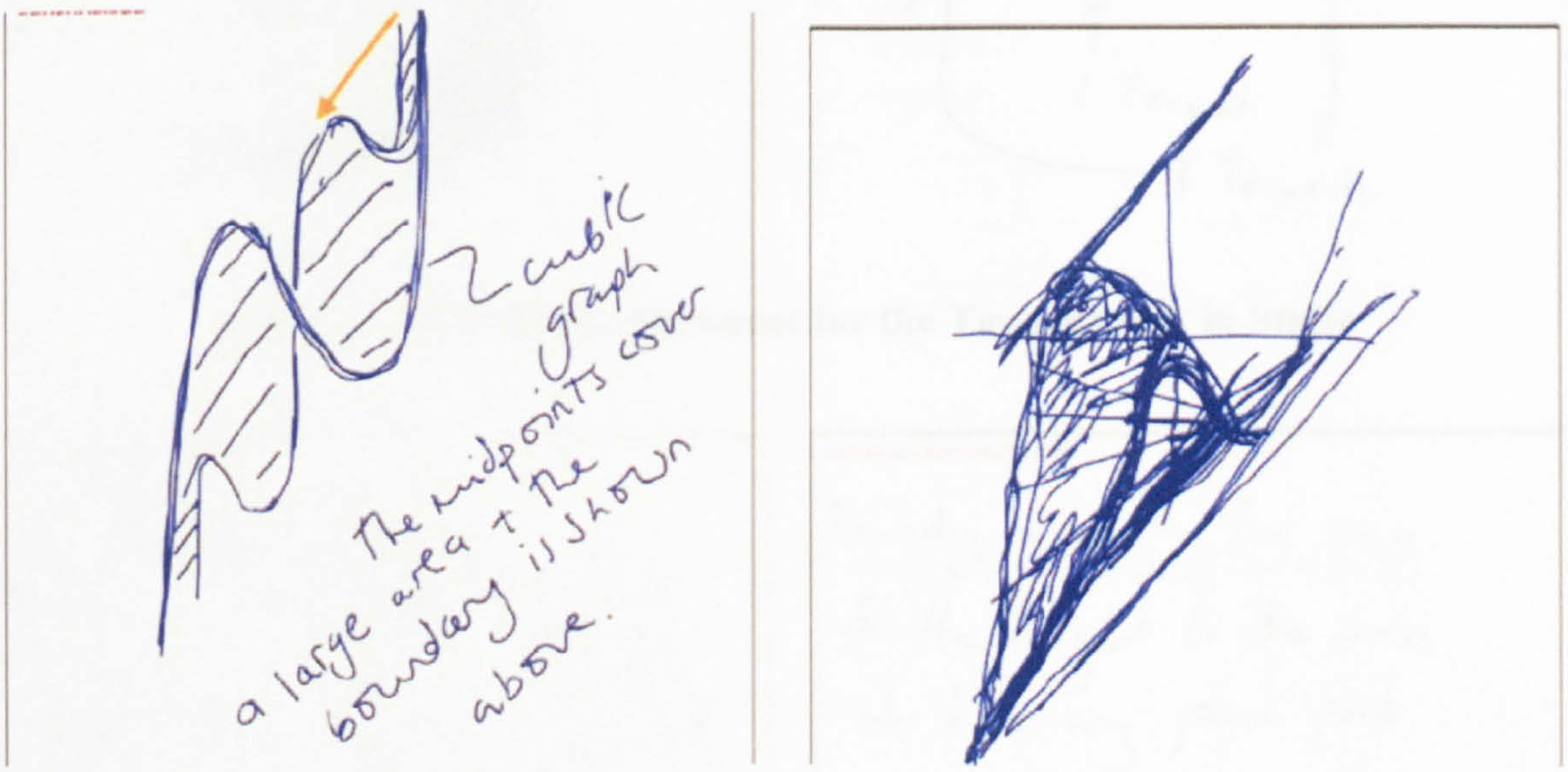


Figure 7-36 Written inferences for Chord task in Interactive

Variation of inferences in the Tangent task between instantiations

For the Tangent task, in Static and in Interactive, the inferences were based on texts and graphs. Only one participant answered the task correctly (Left of Figure 7-37). Similar to the Chord task, the participants related their answer to certain discrete points.

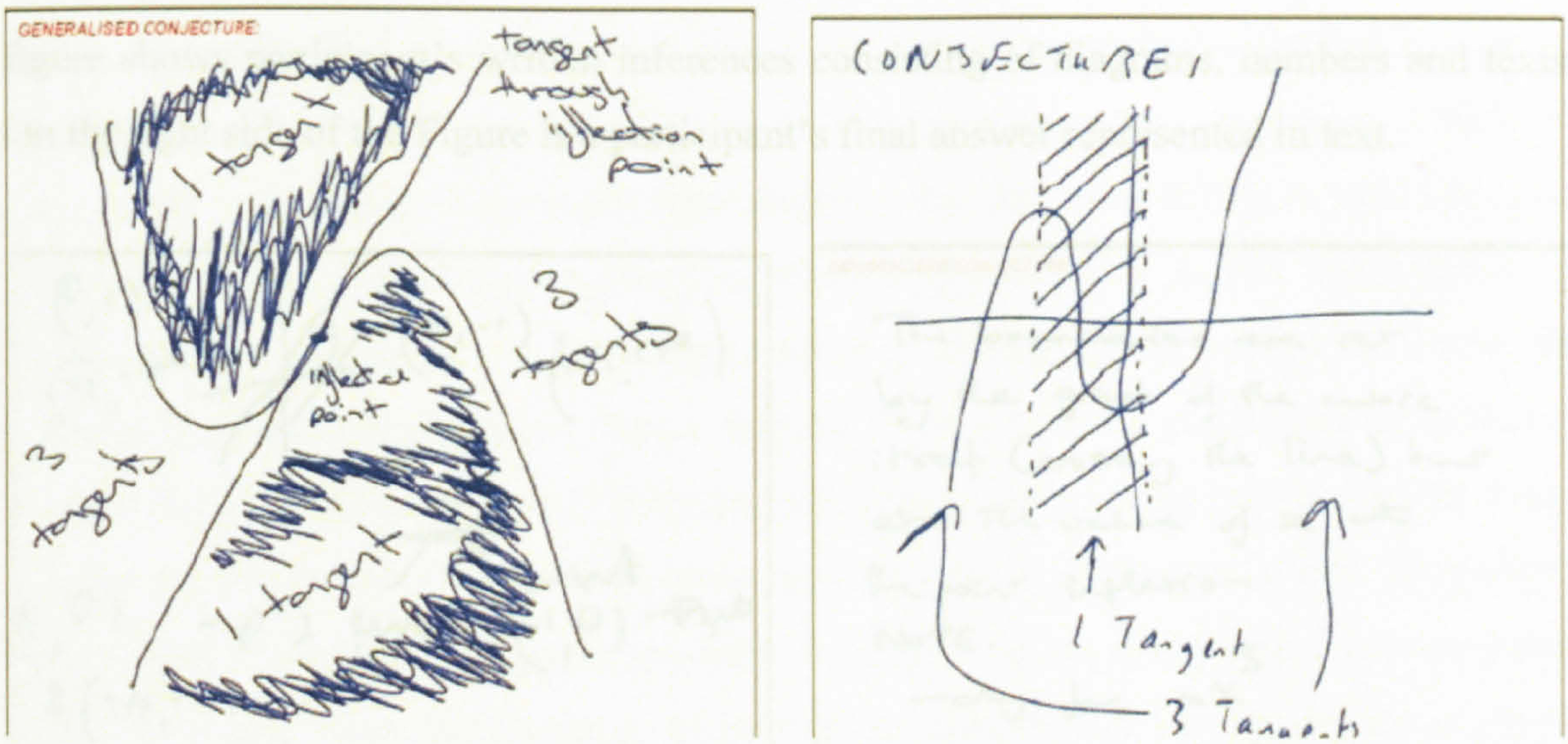


Figure 7-37 Written inferences for the Tangent task in Static

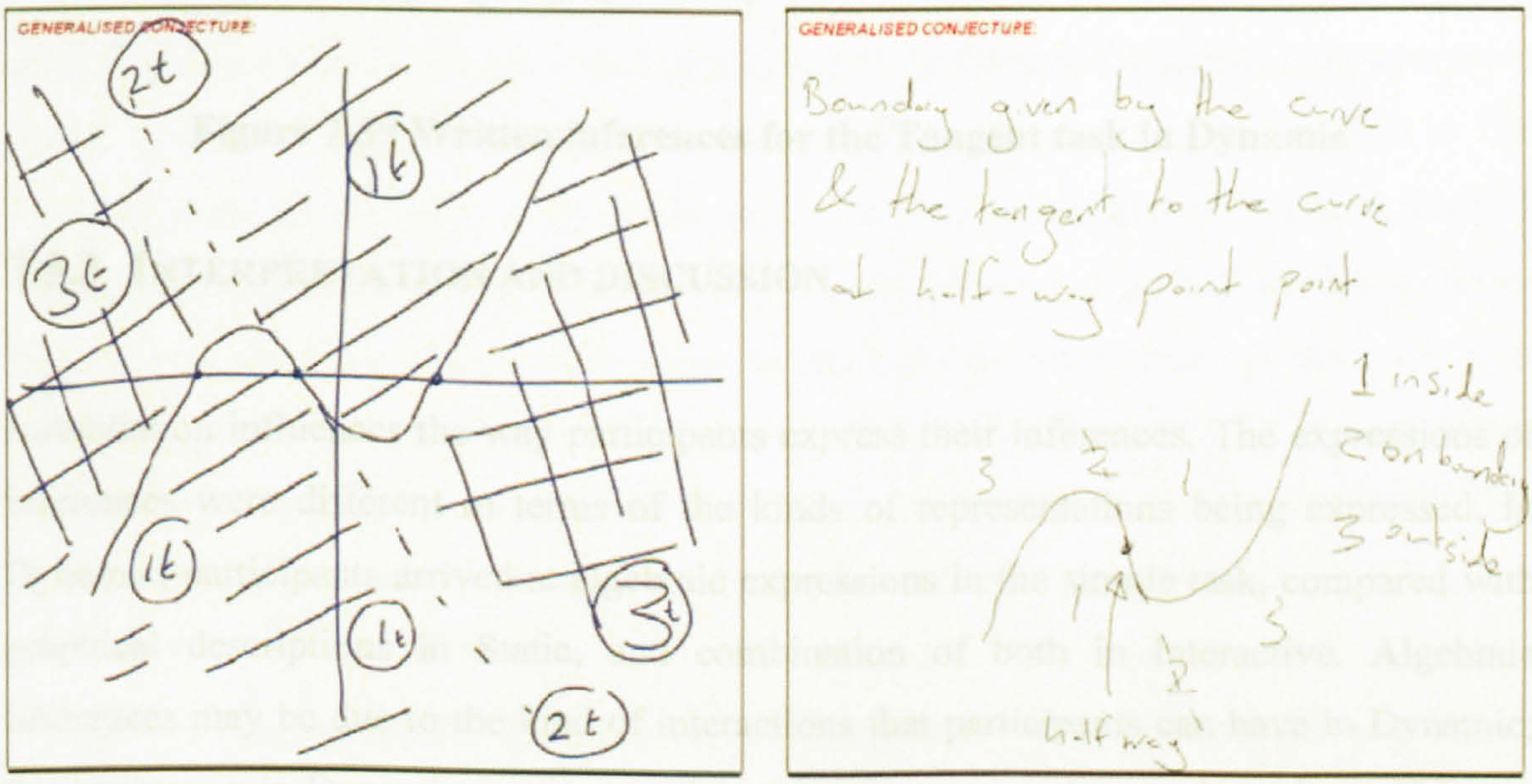


Figure 7-38 Written inferences for Tangent task in Interactive

The written inferences in Dynamic (Figure 7-39) were different compared to non-Dynamic above (Figure 7-38). The inferences were a combination of text, numbers and graphs. Those who got it correct are those who related it to the numbers. Moreover, the participants were able to express their answers textually coupled with graphs. They complemented their diagrams with written explanations. In the example below, the left figure shows participant's written inferences consisting of diagrams, numbers and texts. On the right side of the Figure is a participant's final answer represented in text.

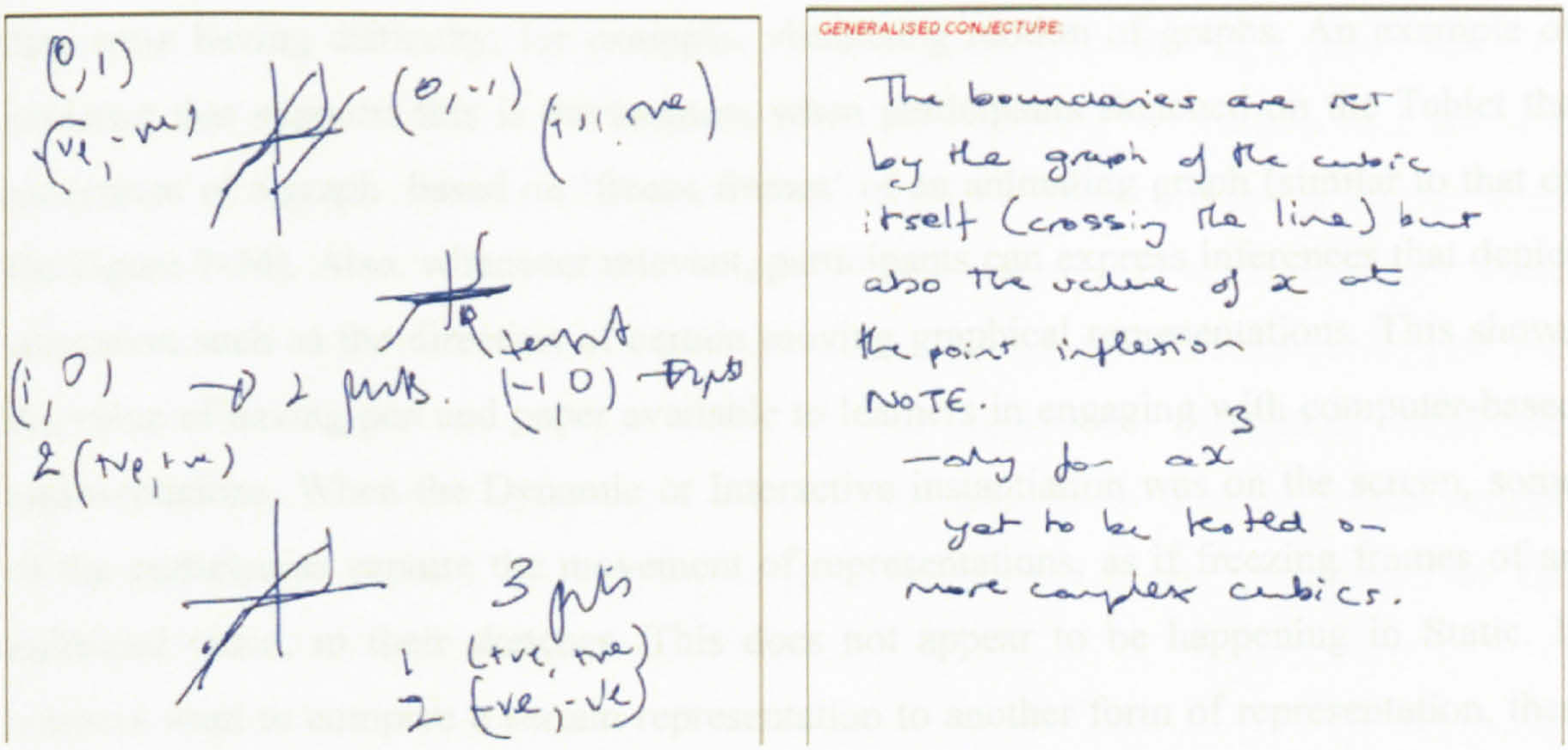


Figure 7-39 Written inferences for the Tangent task in Dynamic

7.5.3 Interpretation and Discussion

Instantiation influences the way participants express their inferences. The expressions of inferences were different in terms of the kinds of representations being expressed. In Dynamic, participants arrived at algebraic expressions in the simple task, compared with graphical descriptions in Static, and combination of both in Interactive. Algebraic inferences may be due to the kind of interactions that participants can have in Dynamic; the interaction in Dynamic requires participants to enter equations and numbers. This can be supported by the algebraic-related strategy found when the instantiation is in Dynamic. This is similar to the point made by Borba and Confrey (1996) that certain instantiations in a computer environment can help learners, for example, to tackle algebraic problems; and when the software does not produce the representation that can test their inferences, learners may construct their own representations.

Another difference found in the expression of inferences between instantiations was in the complex task. Previous research suggests that learners' construction of representations, for example with the use of pencil and paper, can help reduce cognitive load (Cox, 1996). An interpretation of the evidence above is that re-representing strategies reflect the participants' needs to note down the representation that they were trying to tackle because they were having difficulty, for example, visualising motion of graphs. An example of evidence that supports this is the moment when participants sketched on the Tablet the movement of a graph based on 'freeze frames' of an animating graph (similar to that of the Figure 7-34). Also, whenever relevant, participants can express inferences that depict animation such as the direction of certain moving graphical representations. This shows the value of having pen and paper available to learners in engaging with computer-based representations. When the Dynamic or Interactive instantiation was on the screen, some of the participants capture the movement of representations, as if freezing frames of an animated video, in their sketches. This does not appear to be happening in Static. If learners want to compare a certain representation to another form of representation, then doing this in Static could be easier than doing it in Dynamic because they can revisit Static figures by a simple mouse click. In other instantiations, it is not so easy to flick between different displays. In Dynamic, inputs have to be recalled and made quickly. In Interactive, what to drag and where to drag it need to be recalled. The evidence showed that there were fewer occasions when participants noted down their inferences in Static than in Dynamic and in Interactive, (as presented in the Timelines for re-representing strategies in Appendix D). Also fewer participants used re-representing strategies in Static than in the Dynamic and Interactive instantiations, as shown in section 6.3.

Cox (1999) has argued that the construction of graphical representations can help learners to articulate concepts that are difficult to express. The research presented in this thesis supports this view: Participants used a combination of diagrams and words to express ideas that they found difficult to express. Although the conjecture given by P4 does not exactly match one of the expected generalised conjecture stated in section 4.3, her combination of text and diagram provided enough information to suggest that P4 answered the task successfully.

> *P4: I don't know what you call that but I think it is inflection (drawing in Figure 7-39 above) (00:09:45:13)*

Externalisation of representations can take similar or multi-representational form (Cox, 1996). It seems that instantiation can influence the form in which inferences are expressed. In this research, in the simple task, re-representations were in algebraic form or graphical form or a combination of both. The suggestions of Borba, Confrey and Cox above highlight the importance of examining not only learners' interaction with standard external representations but also the representations they create as these latter representations provide information that can be vital in understanding which representation is helping reduce learning problems. This research seems to suggest that difference in interactivity has an effect on the expression of written inferences participants made. In the situations and the evidence presented above, the results suggest either that the participants are actively engaged in processing the information or that cognitive overload is happening (Cox, 1999).

7.6 FINDINGS RELATING TO DIFFICULTIES WITH MULTIPLE REPRESENTATIONS

This section outlines the specific difficulties participants experience in relation to instantiations and external representations.

7.6.1 MAIN FINDINGS

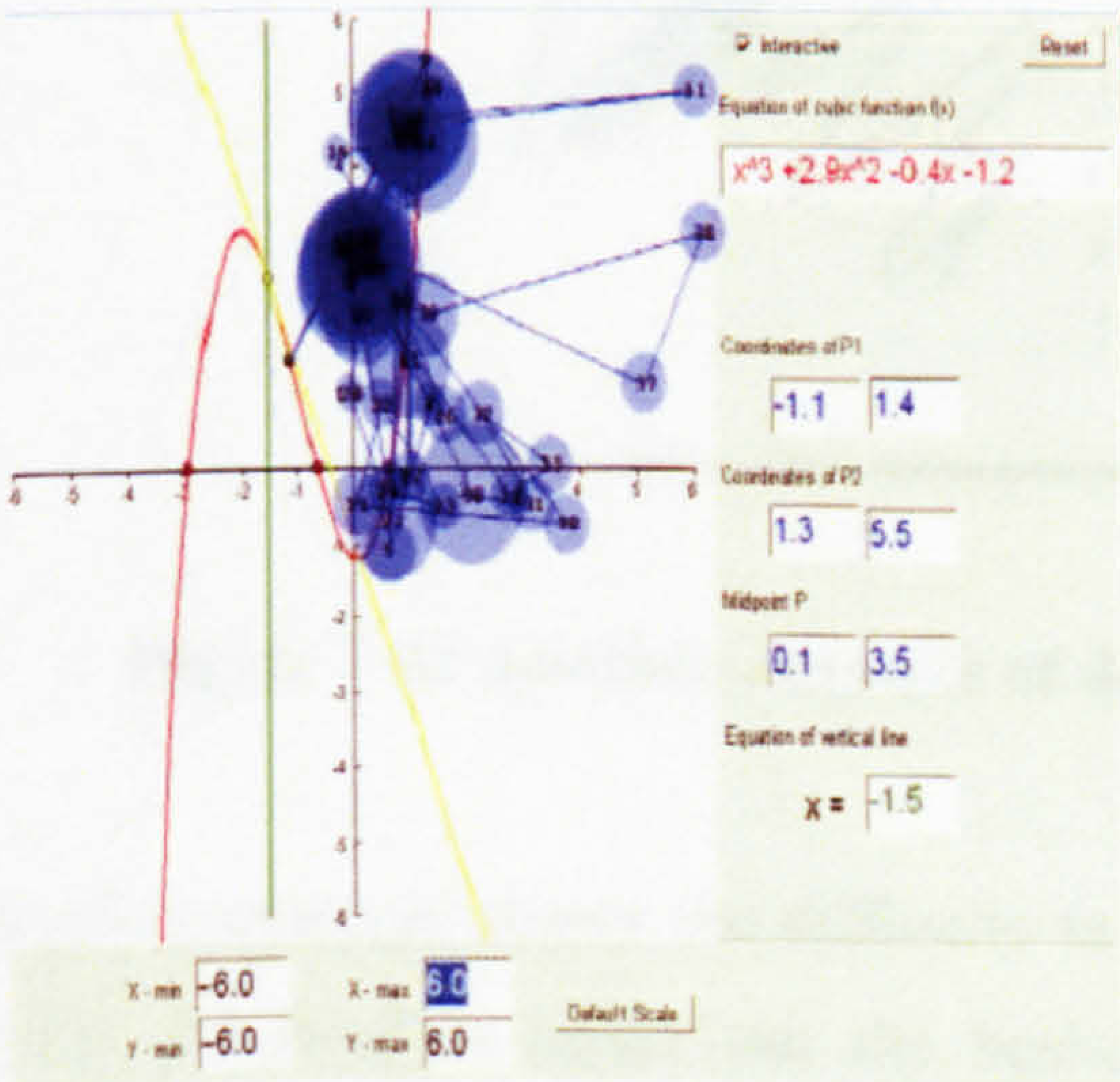
The findings related to difficulties are as follows: the participants had difficulty finding horizontal limits of a function; instantiation could either constrain difficulties related to external representation or could introduce difficulties related to linking representations; in Static, the participants were observed having difficulty searching for comparable representations; in Dynamic, the participants sometimes had difficulty maintaining consistent interpretations of differently scaled graphs; in Interactive, the participants seemed to have difficulty tracing objects diagonally.

7.6.2 EVIDENCE

Analysis in relation to difficulties with external mathematical representations and interactivity is presented below.

Difficulty related to external mathematical representations

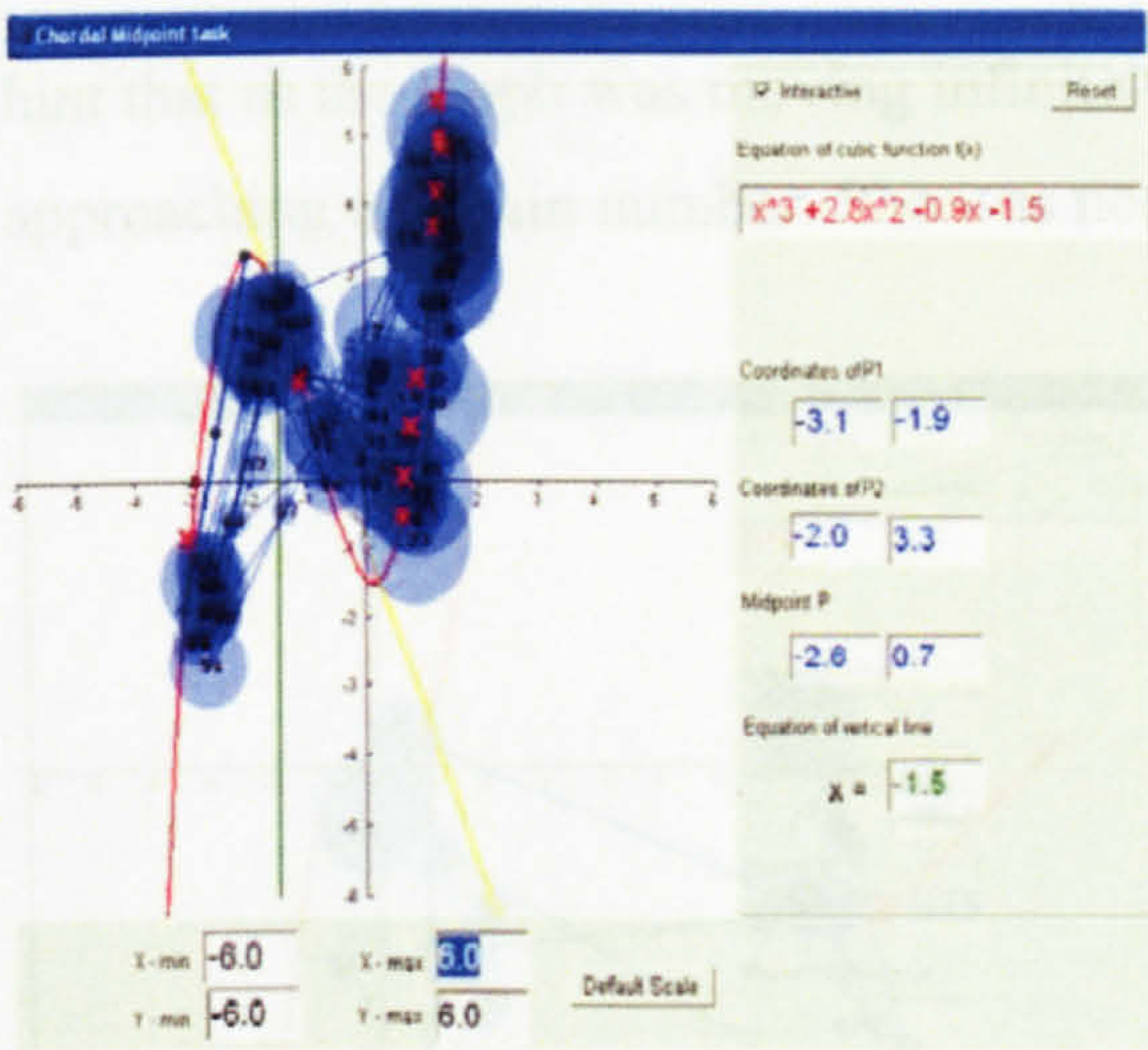
The participants were able to recognise that a certain graph has a vertical limit. The participants were observed having difficulty finding the horizontal boundary of a certain representation. This meant that given the values (x, y), where x gives the horizontal location of a graphical point and y the vertical location, participants can seem to identify the values that relate to the limit along the vertical axis of the graph but not the limit along the horizontal axis. In the given example below, the participant (P10) was able to tell that the graph has no vertical limit (Figure 7-40). He also managed to identify that a graph will follow a certain shape, i.e. stating that a certain graphical point will follow the contour of another graph. This was supported by the gazes tracing the graph (Figure 7-41). However, he also noticed that the graph is bounded horizontally on another side, the left-side of the upper-part of the graph which is symmetric to the right-side of the lower-part of the graph. Figure 7-42 shows that the participant was not successful in finding the horizontal boundary.



00:02:49:09

P10: This point can go to infinity then the... the chord will more or less follow the... the midpoint will more or less follow the chord anyway... so as this point goes to infinity on the x... then the... the midpoint goes to infinity... I cannot say there is any vertical limit where this midpoint will be... although if you think of the boundary... this point is bounded by the cubic... If I make the chord as small as I want definitely the boundary will be the cubic...

Figure 7-40 Finding a graphical boundary



00:04:51:09

P10: It will be the cubic... Err... same thing here.. I can just follow the contour of the cubic... so I will never have a point here... Now how do I know where it stops in here... ok that's a good question... this point could go far to infinity...

Figure 7-41 Difficulty in finding a graphical boundary

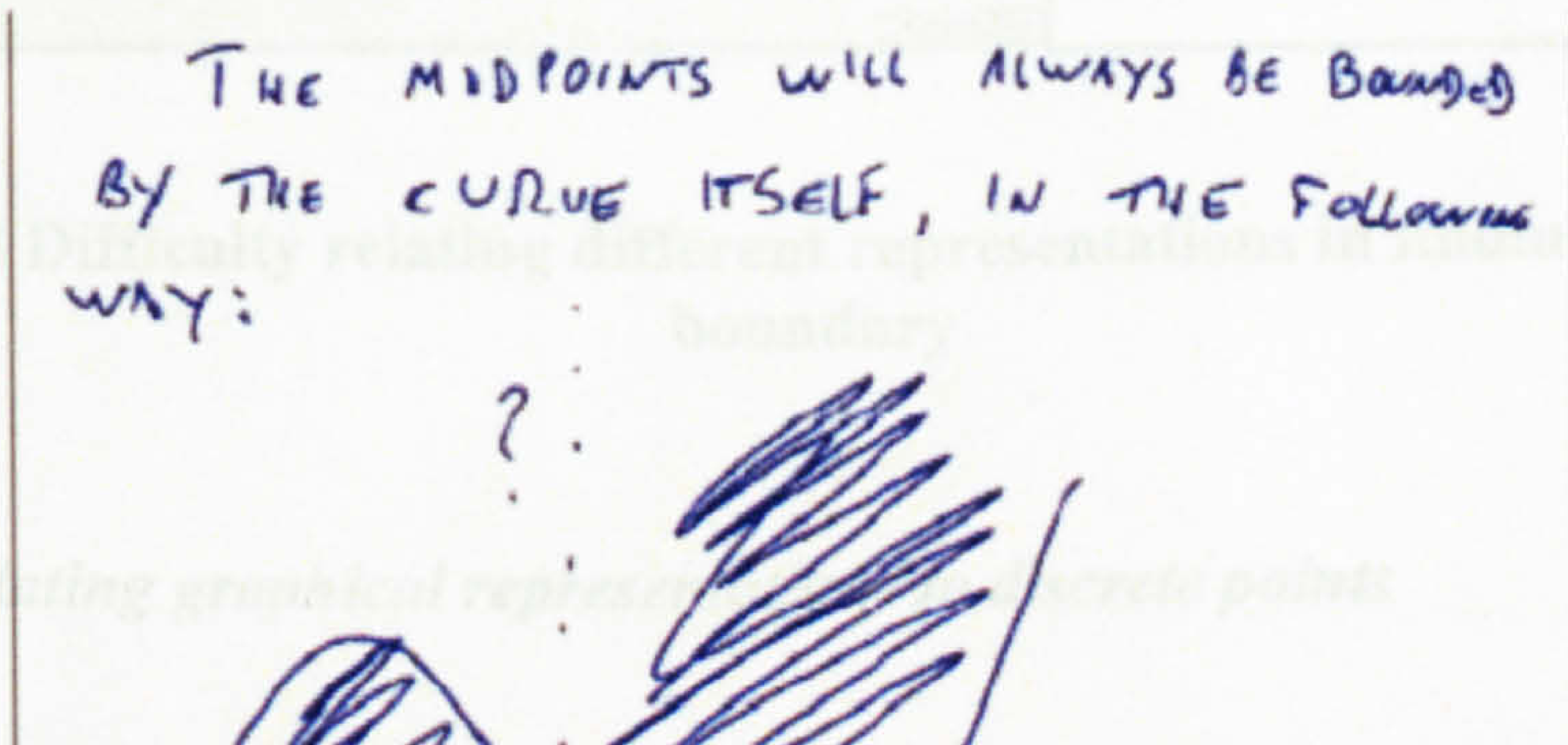


Figure 7-42 Another example of difficulty in finding a graphical boundary

Another example shows the difficulty in finding the horizontal limit of a graph (Figure 7-43). P4 tried to figure out the horizontal limit by looking at the numbers as the animation was playing. This participant was not able to find the limit. However, she had a hint that as the graph was moving infinitely in a vertical direction the horizontal limit was approaching a certain number. She was not able to relate the graph to the numerical limit.

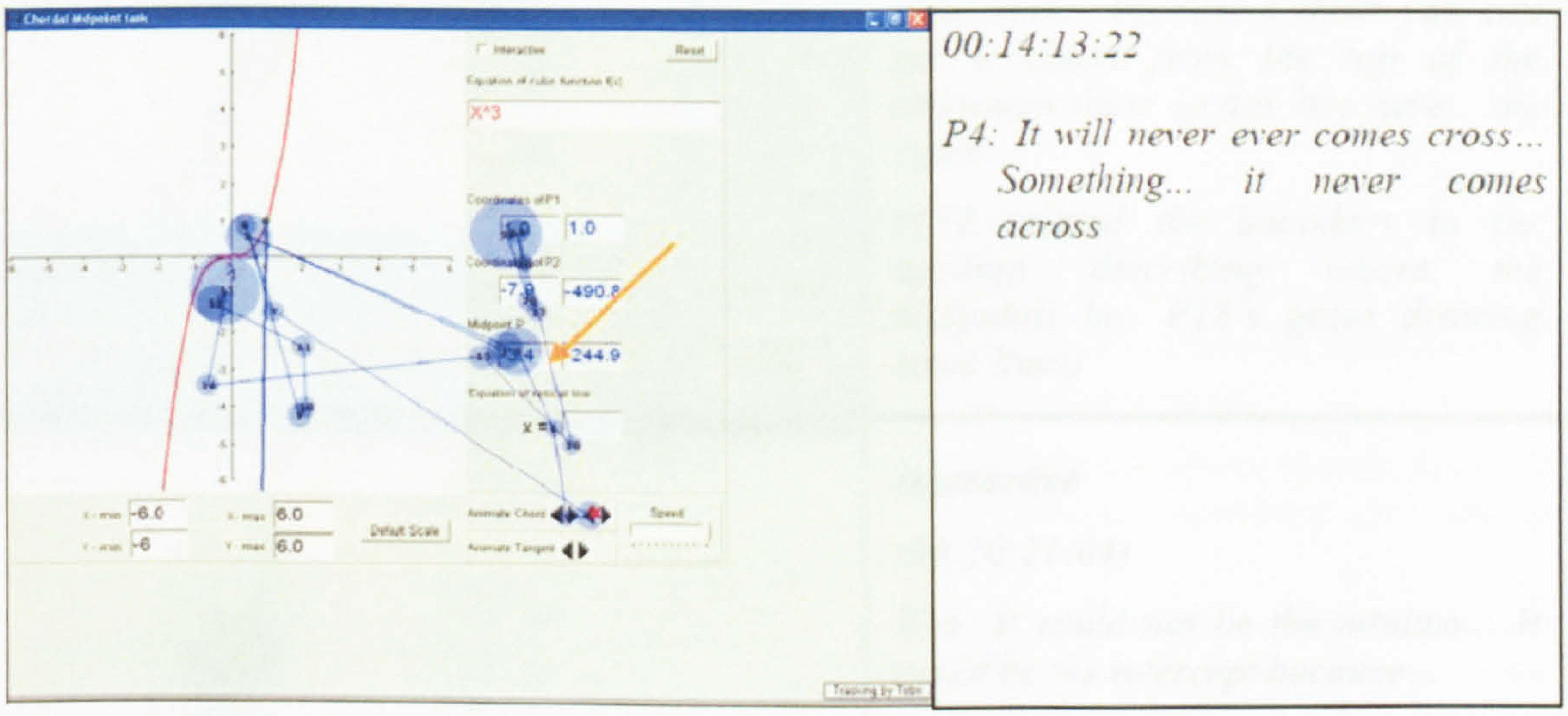
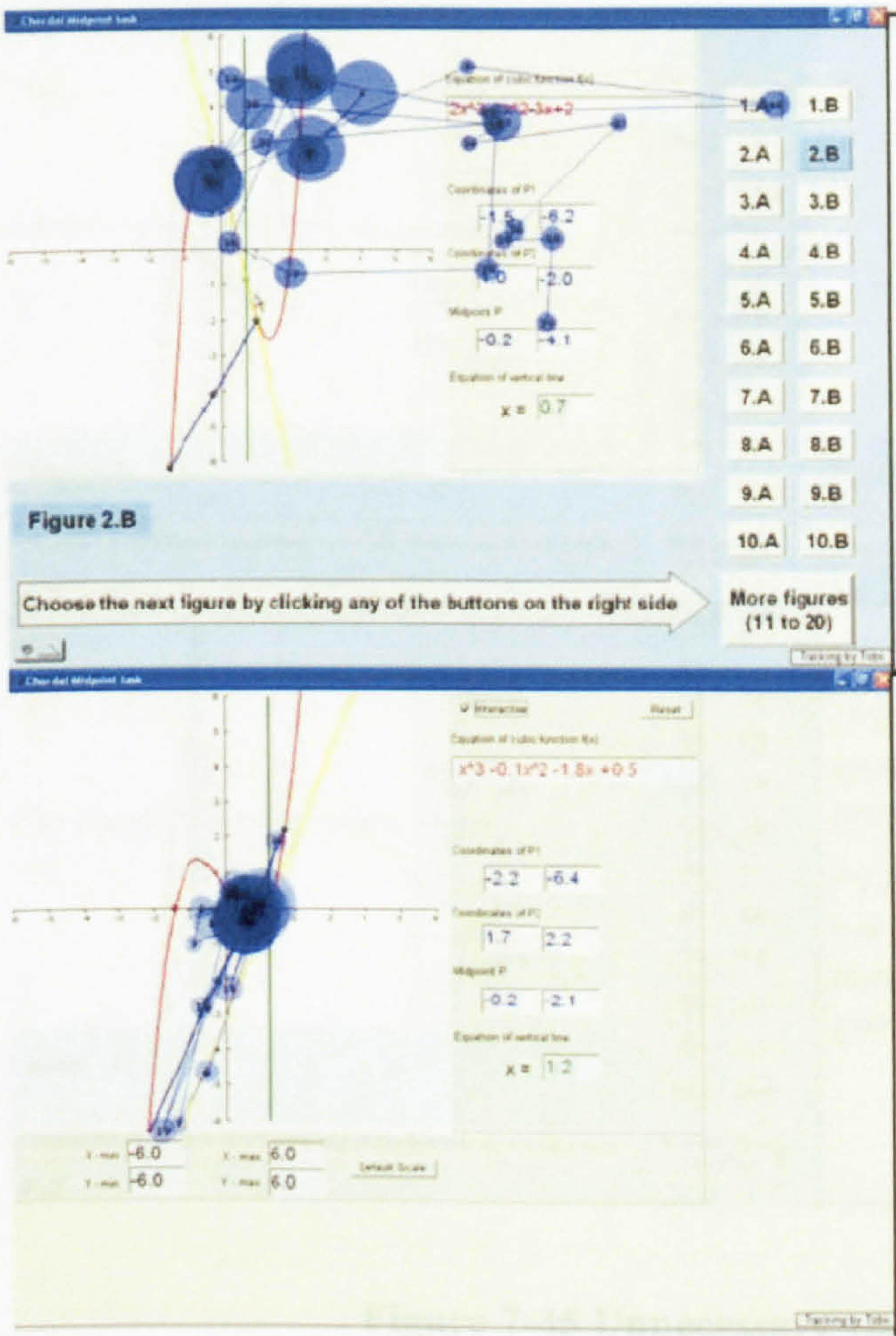


Figure 7-43 Difficulty relating different representations in finding a graphical boundary

Difficulty in relating graphical representations to discrete points

The participants' eye-marks in Static typically depict drawing lines whilst in Interactive eye-marks follow the movements that a point makes. There is a difference in the use of the point-wise strategy between Static and Interactive, in terms of the movement of the eye-marks, before participants make an utterance.

Nobody seemed to use a point-wise strategy in Dynamic. On the other hand, this strategy was used by many participants in Static and Interactive (Figure 7-44). In Static and in Interactive, the participants assumed that 'discrete points' (e.g. the highest point, lowest point, turning point, point of intersection of the horizontal or vertical axes) had something to do with the task. There is a difference found in the use of the point-wise strategy between the Static and Interactive shown by the movement of eye-marks. In Static, the participants' gazes depicted some lines being drawn on the graph whilst in Interactive the eye-marks followed the movement that the graph made as the participants dragged it.



Static

00:05:20:01

P18: You can have midpoints higher than than... because I think you can get a chord from the top of the maximum right up this line here... the cubic...

(P18 related the boundary to the maxima describing where the midpoints lay. P18's gazes drawing some lines)

Interactive

(00:20:21:04)

P16: It could not be the minima... it could be the intercept because...

(P16 related the boundary to the points of intersection of the graph and the horizontal axis)

Figure 7-44 A typical Point-wise strategy in Static

Difficulty with Static instantiations

In Static, participants were observed having difficulty searching for comparable representations in a series of Static pictures. Participants can search for graphs in the series of figures offered in the software. As a result of their search, they can relate the equations and graphs that they found comparable (Figure 7-45).

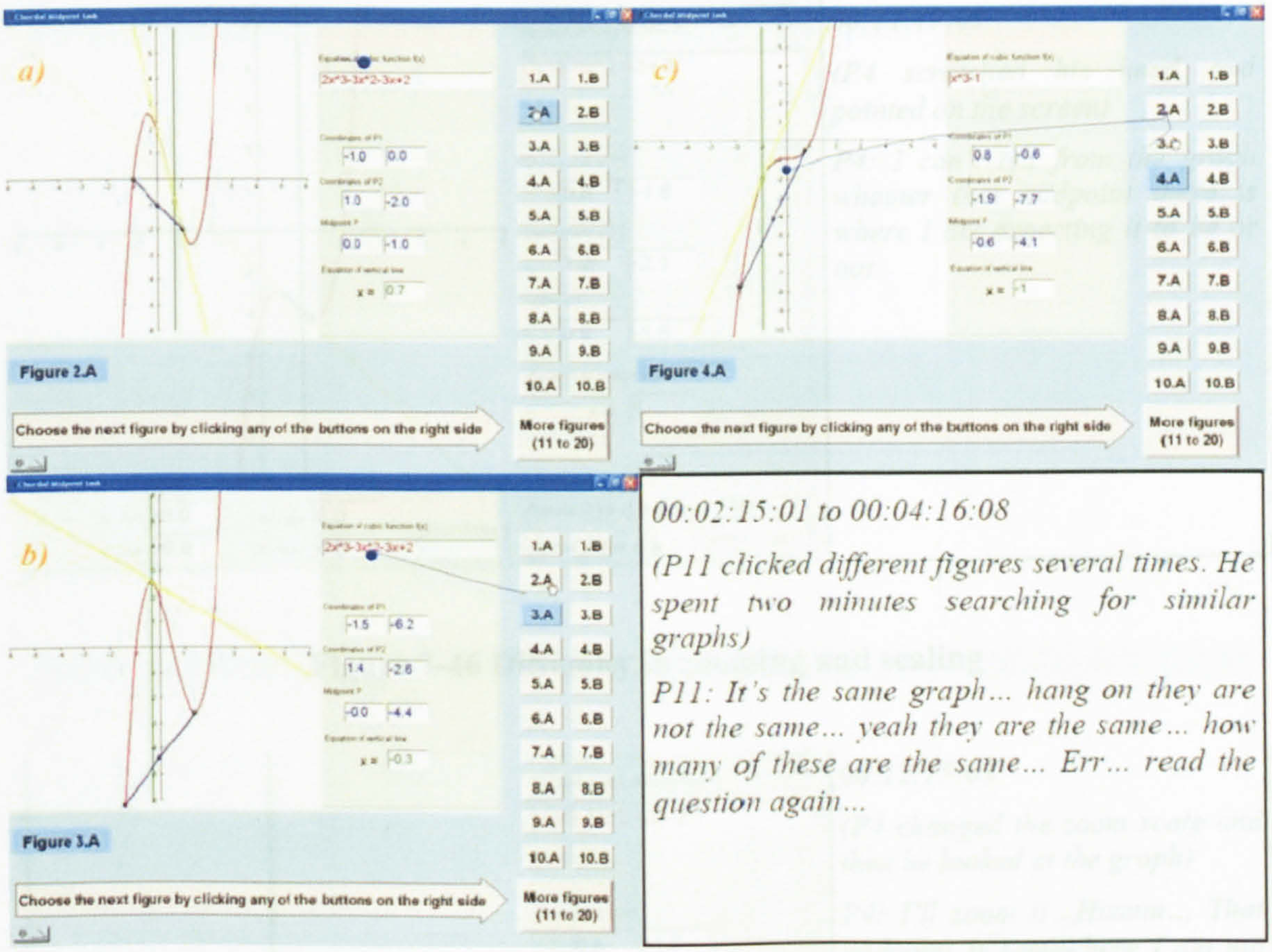
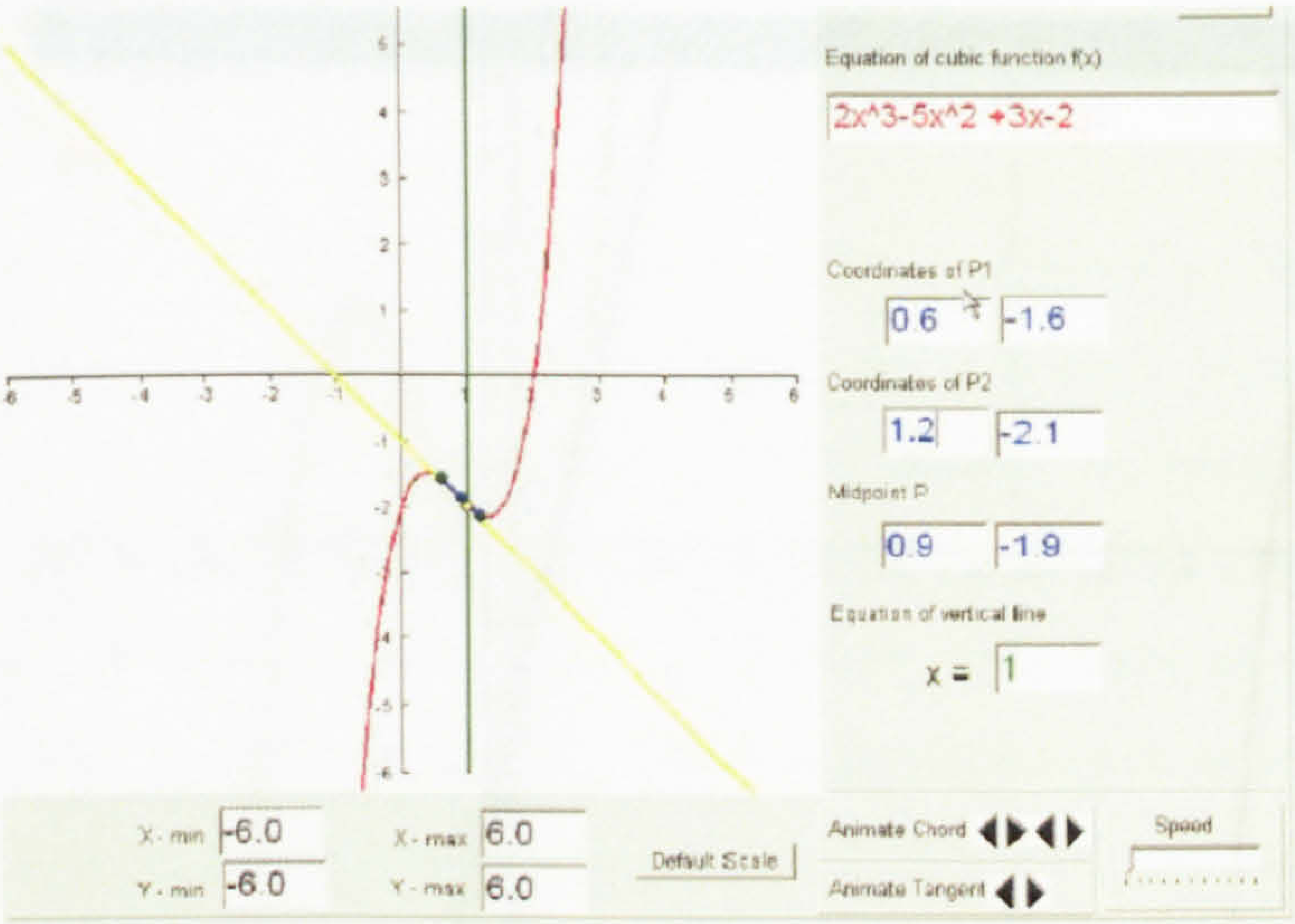


Figure 7-45 Unnecessary visual search in Static

Difficulty with Dynamic instantiations

For Dynamic, when participants zoom out or zoom in they seemed to fail to relate two similar graphic representations of different scale (Figure 7-46).

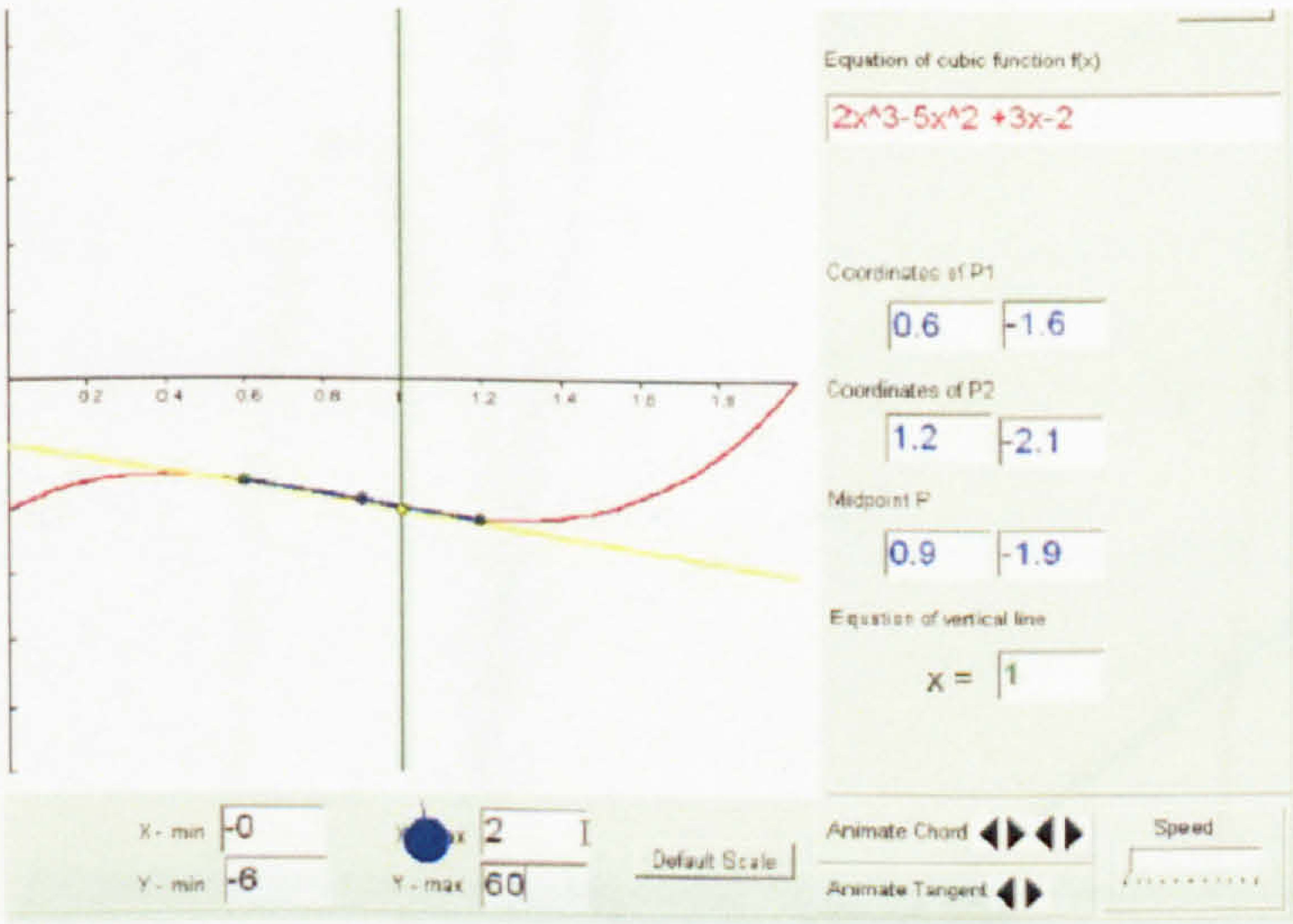


00:12:15:03

(P4 scratched his head and pointed on the screen)

P4: I can't tell from the graph whether that midpoint there is where I am expecting it to be or not

Figure 7-46 Difficulty in zooming and scaling



00:12:15:03

(P4 changed the zoom scale and then he looked at the graph)

P4: I'll zoom it...Hmmm... That midpoint is somewhere I am not expecting it to be...

Figure 7-47 An example of difficulty in making sense of the change in representations

Difficulty with Interactive instantiations

In Interactive, participants were found to drag objects horizontally and vertically but have difficulty tracing objects in an inclined manner. Tracing along the curve where the variable that changes is the x-axis affects the way they interact with the software (Figure 7-48).

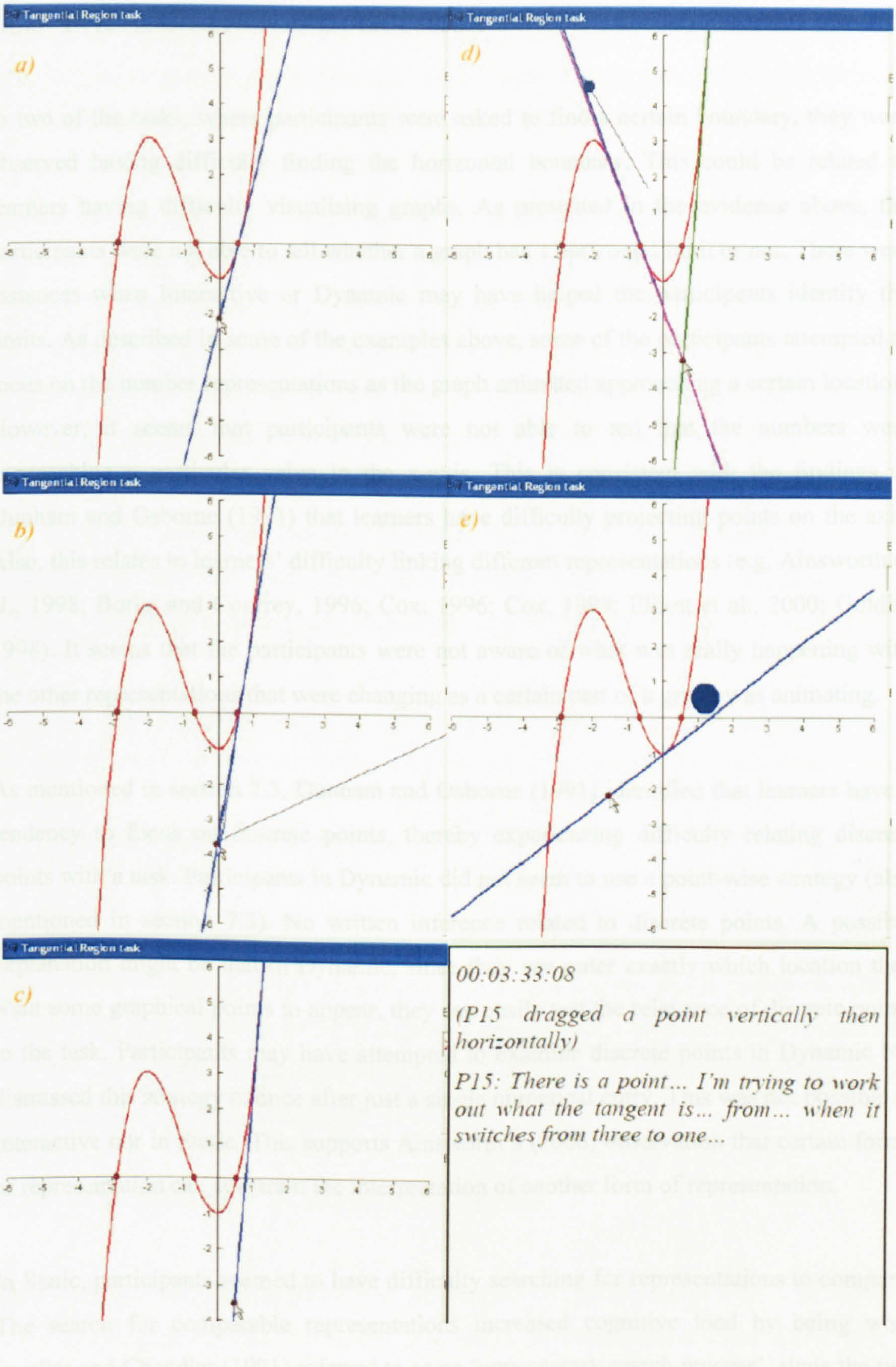


Figure 7-48 Typical directions of dragging objects

7.6.3 INTERPRETATION AND DISCUSSION

In two of the tasks, where participants were asked to find a certain boundary, they were observed having difficulty finding the horizontal boundary. This could be related to learners having difficulty visualising graphs. As presented in the evidence above, the participants were not able to tell whether a graph has a horizontal limit or not. There were instances when Interactive or Dynamic may have helped the participants identify the limits. As described in some of the examples above, some of the participants attempted to focus on the number representations as the graph animated approaching a certain location. However, it seems that participants were not able to tell that the numbers were approaching a particular value in the x-axis. This is consistent with the findings of Dunham and Osborne (1991) that learners have difficulty projecting points on the axis. Also, this relates to learners' difficulty linking different representations (e.g. Ainsworth et al., 1998; Borba and Confrey, 1996; Cox, 1996; Cox, 1999; Elliott et al., 2000; Goldin, 1998). It seems that the participants were not aware of what was really happening with the other representations that were changing as a certain part of a graph was animating.

As mentioned in section 7.3, Dunham and Osborne (1991) identified that learners have a tendency to focus on discrete points, thereby experiencing difficulty relating discrete points with a task. Participants in Dynamic did not seem to use a point-wise strategy (also mentioned in section 7.3). No written inference related to discrete points. A possible explanation might be that in Dynamic, since they can enter exactly which location they want some graphical points to appear, they can easily test the relevance of discrete points to the task. Participants may have attempted to examine discrete points in Dynamic but dismissed this strategy at once after just a single numerical entry. This was not possible in Interactive nor in Static. This supports Ainsworth's (2006) observation that certain forms of representation can constrain the interpretation of another form of representation.

In Static, participants seemed to have difficulty searching for representations to compare. The search for comparable representations increased cognitive load by being what Sweller and Chandler (1991) referred to as an 'unnecessary search process', since the two representations are not found in the same 'slide'. There were indications in the data that participants are attempting to search for representations to compare but it seems that searching and remembering comparable representations from two different slides over-burdened them. They tended not to continue using such a strategy.

In Dynamic, the participants were observed to pay more attention to graphs in detail. This led them to use the zoom functions in Dynamic. However, it was observed that participants were experiencing difficulty relating two differently scaled representations. This is similar to Dunham and Osborne's findings (reported in section 2.2.1 above) that some learners have difficulty in attending to the importance of scales and experience confusion between 'transformation' and 'rescaled' views of graphs.

In Interactive, participants seemed to drag objects in certain horizontal or vertical directions and may fail to drag diagonally. It might be speculated that this is connected with the way people are thought to read graphs and to project points on a Cartesian plane.

The difficulties presented in this section clearly show two types: one relating to the type standard external representation and the other relating to the type of instantiation.

7.7 CHANGES IN STRATEGIES

Some illustrative examples of changes in strategies are given in this section. First, a detailed chronological account of a participant's chosen strategies is presented. This is followed by a comparison of the changes in strategies of two participants undertaking another task. Finally, an example is given of a participant's different strategies in two similar tasks of varying instantiation.

7.7.1 A PARTICIPANT'S CHOSEN STRATEGIES

A complete verbatim transcription with screenshots that details the coding of strategies is presented in section 5.4.2. This example is that of P4's completion of the Chord task in Dynamic instantiation. The Timeline below (Figure 7-49) shows P4's chosen strategies. It also shows the representations that P4 looked at during the course of the task. The completion of the task is divided into nine interesting episodes (marked as i, ii, iii…, ix). Each of the episodes represents illustrative situations with which P4 changes strategies related to the findings presented above. A description of what actually transpired in each of the episodes is given and is accompanied by the utterances. Each episode is interpreted to illustrate possible explanations on the changes in P4's strategies.

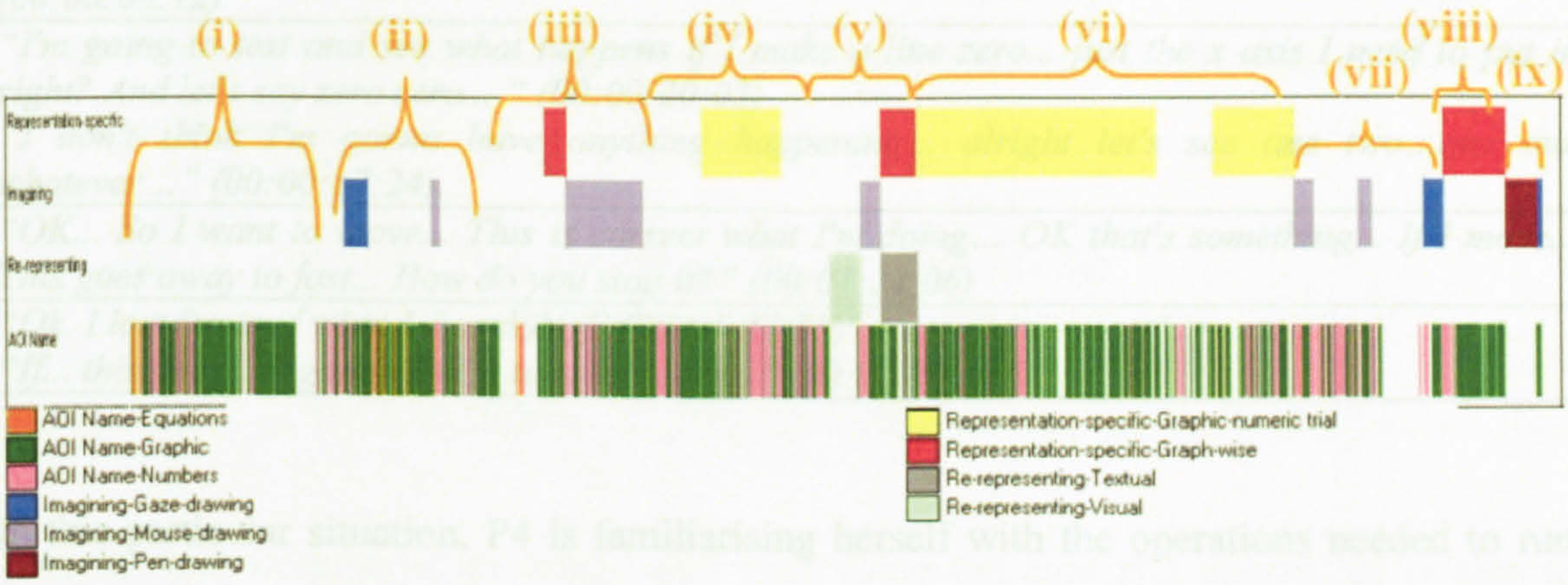


Figure 7-49 Strategy used by P4 in Timeline visualisation

The Chord task is: *"A chord can be drawn on the curve of a cubic function. What can you infer about the midpoints of the chords drawn between two points on any cubic function?"* The expected answer is "a set of all points bounded by the axis and the cubic itself."

P4's final conjecture is given in Figure 7-39. The conjecture is 'correct'. This conjecture does not exactly match the expected generalised conjecture as expressed in 4.3 but the sketch given by P4 with description has the same meaning as the stated expected generalised conjecture.

Getting familiar with the software

P4 starts figuring out what the software can do. She enters an initial equation to start with, which she refers as 'simple'. She enters other numerical inputs and plays the animation.

"Again I always like to do simple stuff so I gonna go x^3 and see what happens there..." (00:00:08:12)
"I'm going to test and see what happens if I make a line zero... just the x axis I need to put in right? And let's say zero zero... " (00:00:20:02) *"I don't think I'm gonna have anything happening... alright let's see one two...one one whatever..." (00:00:37:24).*
"OK... So I want to move... This is correct what I'm doing... OK that's something... If I move... This goes away to fast... How do you stop it?" (00:01:24:06)
"Ok. I lost focus of what I was doing" (00:01:31:05) *"If... this goes... It gonna find a tangents surely" (00:02:09:11)*

In this particular situation, P4 is familiarising herself with the operations needed to run the software. P4 is keeping track of the changes with the representations given on the interface. She is observing the feedback that the software does as she inputs equations and numbers and as she plays the animation of the software.

Indications of difficulty

P4 gazes show that some lines are being drawn. P4 tries to make sense of the graph as she draws some lines using her eyes. She uses the mouse to draw more lines by moving the pointer from a point on the graph to another point along the graph.

gaze-drawing	*"I'm not certain what this means..." (00:02:37:24)*
	"minus 1... I'm just checking to see... which is the next point I should look at... which is minus one..." (00:02:57:00) *"Err... Let me see what happens when I use... a tangent line... a vertical line... x equals zero is the easiest... (Inaudible)" (00:03:14:02)* *"Oh that's a tangent line... ok... equation of the vertical line..." (00:03:27:19).*
mouse-drawing	*"That's a tangent line... ok... equation of the vertical line..." (00:03:40:17).*
	"I don't have a clue what's going on...I'm getting lost with this question..." (00:04:34:00)

P4 seems to plan what number or equations she is going to input. P4's talk seems to suggest that she is experiencing difficulty. She is trying to recognise the location of the 'midpoints' by drawing some 'invisible lines' using her eyes (as shown by the scanpaths) and using the mouse.

Graphical description

P4 describes the movement that the graph makes. She keeps on drawing invisible lines using the mouse.

graph-wise	*"Yeah... I know that thing... but..." (00:05:04:05)* *"The midpoint goes up... it goes up..." (00:05:17:23)*
mouse-drawing	*"Let me just try something easy... (inaudible)... I can stop that... Zero zero that's the easiest... It's the region that moves with this" (00:05:19:17)*

After several attempts in drawing some lines using the eyes and mouse, P4 is able to describe the behaviour that the midpoint makes. After describing the behaviour that the graph makes and more attempts of drawing using the mouse, it appears that P4 is going to try a different approach.

Attention shifts

P4 decides to play and watch the graph's animation. P4's gazes show that she is paying attention to the numbers based on scanpaths. She is also looking at the graph. P4 makes an inference about the half part of the graph as the animation is playing. P4 then tests this inference with the other half part of the graph.

	"It goes from zero... it goes to the tangent... zero zero... and decreases" (00:06:45:09) *"I'm just testing to see if the midpoint ever crosses... if ever crosses line here" (00:06:58:18)*
graphic-numeric	*"If I increase the speed maybe I'll know... I don't know it looks... A bit likes... asymptotic to it... (Inaudible) ...whether it's gonna be asymptotic to it... it is not moving out... right..." (00:07:50:19)*
	"That's possibly something there... let us try one... I expect the same thing should happen. It's really asymptotic here... that's alright... err... So that's the point zero zero... if I have it on... zero zero and... and then one... (Inaudible)... If I do it from minus one..." (00:08:26:16)

P4's decision to watch the animation shows that P4 pays attention to numerical representations. P4's scanpaths show that she spends more time looking at the numbers than the graph (see the Timeline above). P4's attention seems to shift from the graph to the numbers. It appears that P4 manages to come up with an inference based on numbers and graphs by changing the focus of attention.

Captured movement on paper

P4 starts sketching on the Tablet PC. She draws a cubic graph and indicates some arrow diagrams and text denoting "asymptotic". She uses the mouse to draw some lines. Whilst P4 is sketching, she also gazes back a few times on the screen. P4 then describes the movement that the graph makes. Then she writes some text on the Tablet PC denoting "tangent 0".

visual	*"Let's assume something... If I have a line from here to here and moving this way it goes asymptotic... asymptotic... If I have it this way zero zero to here it goes this way it is also asymptotic..." (00:08:32:20)*
	(left part of the worksheet) A cubic graph with an arrow pointing towards the left. Wrote "asymptotic" *(Right part) A cubic with an arrow drawn going to the right. Wrote "asymptotic"*
mouse-drawing	*"let me try something... zero zero and... it move just this way... let's move down" (00:08:55:10)*
graph-wise	*"So all it does... So all it does... comes down comes down... until it becomes tangent line zero and it changes direction... yeah..." (00:09:10:05)*
textual	*Drew an arrow pointing left and wrote "tangent 0"*

In this episode, P4 expresses an inference using a combination of visual representations and text. P4 seems to capture the movement that a graph makes by using some diagrams. It seems that P4 keeps an actual record of the behaviour of the graph. This description depicts the movement of the graph during animation. P4's sketches seem to emulate her observations in *(iv)*. During sketching, it seems that P4 tries to re-enact the animation by looking at the graph on the screen. P4's scanpaths are found on the same area where the animation of the sketched graph was.

Shifted attention

Again, P4 enters another numerical input and plays the animation. P4's gazes are observed following the midpoint as it animates. P4 gazes on the numbers corresponding to the midpoint. P4 then tries to make another inference.

graphic-numeric/	*"Alright... Ok... and then if I used one to minus one... (Inaudible) (So it just moves along) Ah! Ok...using minus one to one... It goes down that way... So it crosses the line... But... it cuts one zero..."* *"So that's down right...and it still keeps going...How long can this go to?... I'm still trying to figure out what's going on...That keeps going... that just never...When does the midpoint end..." (00:09:37:00)*
graphic-numeric	*"It gets to... who cares... minus six let's go back to that... This is minus...Tangent is here... it goes up... so obviously... it's doing that... I understand that one... zero... it would draw a line there... It is going to fast... It reaches the tangency... It goes down... it goes down... it reaches zero and then... let's stop there... The midpoint is minus zero point nine minus ten point five..." (00:13:14:00)*

P4's scanpath, in this episode, suggests that P4's attention is focused on a particular part of the graph and the corresponding numbers. P4's gazes also suggest that she is ignoring some of the other representations that are changing during animation. It seems that by focusing her attention to a particular movement of a part of the graph, P4 is having difficulty trying to come up with an inference.

Visualising movement

P4 stops the animation. She then draws some lines using her eyes and the mouse. P4 is trying to make another inference. She restates her previous inference expressed in (v).

mouse-drawing	*"It will never ever comes cross... Something... it never comes across..." (00:14:13:22)*
	"Ok let us try something else... three zero... In fact, two... It is doing it again... one point five..." (00:14:27:18)
mouse-drawing	*"One point five and..." (00:15:01:03)*
	"OK... What conclusion can I make from this... I know what's happening when the tangent is zero... when the tangent is zero... All you have is that the midpoint keeps going this way until it's asymptotic to that axis... And then we have the next one... it is asymptotic to that axis... when we do... but then it passes through that point... and then the same thing happens here..." (00:15:47:10)
gaze-drawing	*What will happen is that... it goes through..." (00:15:49:00)*

P4's scanpath and mouse-movement seems to depict the same movement that graph makes. P4 manages to make an inference related to the lower-part and the upper-part of a cubic graph. P4 seems trying to make a final conclusion.

Indication of difficulty

P4 is trying to make a final conjecture. She describes the behaviour of the graph.

graph-wise	*"It goes to zero zero... It then goes through... It becomes a tangent to it... It shifted... it shifts... and then it becomes asymptotic to something which I can't figure... but it is... Is it the point itself one zero... It could be the point itself one zero... because it never ever comes across" (00:16:03:17)*

P4 seems to be experiencing difficulty. She cannot figure out the 'boundary' related to what is being asked in the task.

Making the final conjecture

pen-drawing	*(Right-most part of the Tablet PC) Drawing line from (0, 0) going upwards. (Bottom part of the Tablet PC)* *Drew an XY plane. Mouse pointer making some lines. (00:16:48:05)*
gaze-drawing	*00:17:11:01*
	"Uhm... I'll make a conclusion I know it's wrong but I'll make it... because I really don't know what is going on now... Alright... let's see what's the inference from this... The chords... the midpoints... of the chord... are bounded... No. wait I need to check something... I did zero one... I did one not... It goes... blah blah blah blah... Ok yeah it seems... are bounded by the bounded by the points... what will I say... by the... No." (00:18:50:05)

P4 sketches on the Tablet PC. The stylus pointer shows invisible lines being drawn before actually making the mark on the Tablet PC. P4 also gazes back at the screen here gazes also show that lines are being drawn.

Before P4 makes her final conjecture, she attempts to draw using the pen and her eyes. It seems that during the time P4 said *"she needs to check something"*, P4 is using the screen to recall the movement of the graph based on a particular input made.

7.7.2 COMPARING CHANGES IN THE STRATEGIES CHOSEN BY TWO PARTICIPANTS

This section illustrates differences in the choice of strategies of two participants in a given task. The task is:

Root task: "An original cubic function f(x) is rotated 180 degrees about a point (a, 0). What can you infer about the solutions of the new function?"

The expected generalised conjecture is:

"The new roots are reflections of the roots of the original. The same distance from point (a, 0)."

P15's answer was expressed in algebraic form (i.e. y = 2a – x) which is considered 'correct'. P2's answer was also correct expressed in text form (i.e. the solutions of g(x) are reflections around x = a of the function f(x)).

The participants are P15 and P2. The task is completed in Static. The strategies and shifts of attention are presented in the Timelines below (Figure 7-50 and Figure 7-51). Only a summary describing the actual events are given in this section and followed by the interpretation. Examples of detailed transcriptions are given in appendix G.

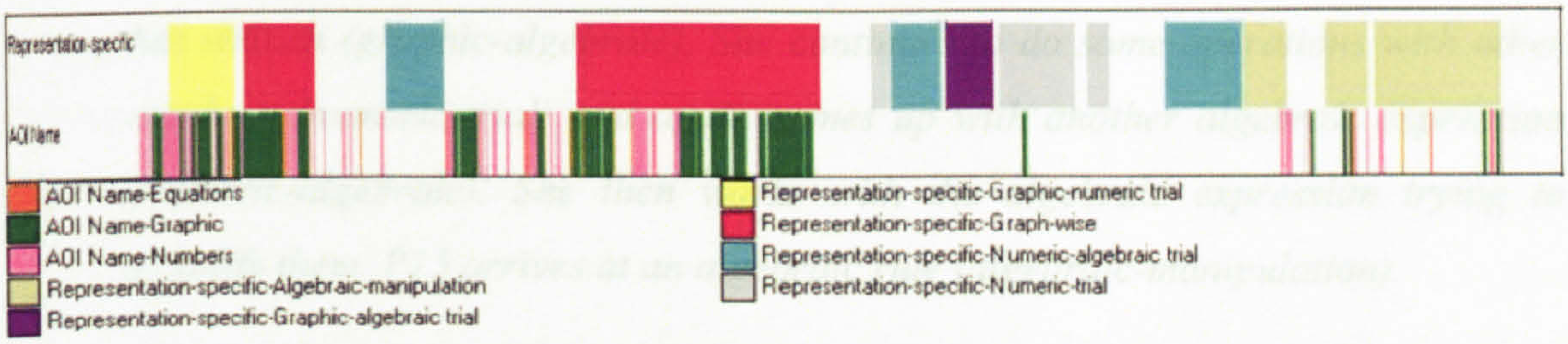


Figure 7-50 Strategy used by another participant (P15) in the Root task

Below is a summary of P15's approach:

> ***(i) Browses the Static figures given:*** *P15 starts browsing two of the Static figures given. She looks at the cubic graph on that first figure; and then she looks at the numbers on that second figure (codes as "graphic-numeric"). Afterwards, she makes an inference that she "knows what the rotation means"; and describes the location of the graphs (coded as "graph-wise"). At this point, not many eyemarks are found on the equations.*

> ***(ii) Attempts to work on paper:*** *She then starts writing in the Tablet PC. She operates on the numbers and makes some approximations. She comes up with an algebraic expression "2a" denoting that the numbers are being rounded (coded as "numeric-algebraic").*

(iii) Compares what is written with what is on the screen: *P15 then looks back at the graph on the screen and compares what she has written on paper and what is on the screen. P4 makes an inference about the distance of the two graphical points she can see on the screen (coded as graph-wise)*

(iv) Dismisses working on the screen: *P15 then works with the Tablet PC. She did not gaze at the screen for quite a time. She operates on the number (numeric-trial) and comes up with an algebraic expression (numeric-algebraic). She then looks at the screen for a short time and relates the graph with the algebraic expression she has written (graphic-algebraic). She continues to do some operations with other numbers (numeric-trial) and again comes up with another algebraic expression (numeric-algebraic). She then works with the algebraic expression trying to simplify them. P15 arrives at an algebraic rule (algebraic-manipulation).*

(v) Uses the software to verify conjecture: *P15 then tests the algebraic rule by choosing a figure on the screen. P15 generalises her conjecture.*

P15 was identified in the previous sections 7.2 of this chapter to be atypical in the way she completed the Root task in Static because she focussed attention on developing an algebraic rule on the Tablet PC. It would be that this participant has a strong preference for algebraic representations.

When looking at the screen, P15's attention is centred more on the numbers than on the graph. The first set of strategies chosen is graphic-related (graphic-numeric and graph-wise). Then, there is an attempt to make sense of the numbers and the graph separately. She is able to establish the connection between the distance of the graphs and how it is reflected in the corresponding numbers given. P15 then continues using strategies related to numbers and algebra. It seems that she is able to arrive at an algebraic rule after having successfully linked the three representations. The software is used as if it were a blackboard. However, the software is also used in checking the final answer.

Figure 7-51 Strategy used by a participant (P2) in the Root task

Here is another example of a participant's completion of the Root task in Static. This participant (P2) also dismisses working on the screen. A summary of P2's chosen strategy is presented. This is also given in the Timeline above (Figure 7-51). A full verbatim transcription is in Appendix G.

> *P2 starts familiarising himself with the software. He looks at the graph, equations and numbers on the screen. He notes some numbers down and decides to work mathematically. He attempts to make a connection between the graph on the screen and the numbers he has noted down (graphic-numeric). He then decides to browse more Static figures. Afterwards, he mentions that he has misunderstood the task. He then has tried to stop himself looking at the screen; but immediately looks back on the screen. He describes the position of the points (graph-wise) and mentions his plans on how he is going to approach the problem mathematically. He then chooses a coordinate of a point on the screen. He compares the numbers with its corresponding graphical equivalent (graphic-numeric). He then makes an inference about the graph (graph-wise) stating that the "solutions are reflected around a point." He then operates on the numbers he has written on the Tablet PC. He spends a long time operating on the numbers. He ends up making an inference about the distance and the numerical coordinates (graphic-numeric).*

In P2's situation, he explicitly states his preference to work with algebraically. P2 even attempts to ignore the screen. However, unlike P15, it seems that P2 is not successful in linking the representations given in the task. P2 answers the task correctly using graphical representations. P2 seems to perceive a graphical approach as inaccurate. It is presented in the evidence that P2 arrives at a correct answer by managing to describe that the "solutions are reflections around a point". Yet, P2 insists on doing the task mathematically. He is not able to pursue an algebraic strategy because he fails to find a pattern in using a numeric strategy.

There are differences in the strategies chosen by P2 and P15. P2 chooses strategies related to graphs and numbers. Although he attempts to use an algebraic strategy, he fails to do so. It seems that P2 tries to mentally find a pattern. This is due to the absence of the computations on the Tablet PC. Unlike P15, she makes the computation on the Tablet PC; and is able to compare the different computations. It seems that both P2 and P15 have a preference for algebraic representations. The times they spend looking at each of the representations are different. P15 spend more time looking at the numbers than P2; and P2 spends more time looking at the graph than P15. Further research could usefully look at whether such differences results in different kinds of inferences.

7.7.3 CHANGES IN STRATEGIES OF PARTICIPANTS IN TWO TASKS

The Chord and the Tangent tasks are similar in some ways: they are similar in that the average time participants spend with the task are similar; the fact that participants use mostly graphic-related strategies; and the similar use of imagining and visual re-representing strategies.

Table 7-2 shows the representation-specific and representing strategies chosen by single-instantiation participants in the Chord and the Tangent tasks. When the instantiation is the same, the strategies chosen in one task are not that different to the strategies chosen in the other similar task for each participant.

Table 7-3, on the other hand, shows the representation-specific and re-representing strategies chosen by varying instantiation participants. The strategies chosen by each participant between the two similar tasks of different instantiation appear to differ.

Table 7-2 Strategies chosen by single-instantiation participants in the Chord and Tangent tasks

	Representation-specific						Re-representing		
Participant Task	Graphic-algebraic	Graphic-numeric	Graph-wise	Numeric-algebraic	Numeric-trial	Point-wise	Symbolic	Textual	Visual
P1 SC		✓	✓		✓	✓			✓
P1 ST			✓			✓			✓
P2 SC		✓	✓						✓
P2 ST			✓						✓
P3 DC			✓					✓	
P3 DT		✓				✓			✓
P4 DC		✓	✓					✓	✓
P4 DT			✓					✓	✓
P5 IC		✓	✓			✓			✓
P5 IT			✓			✓		✓	✓
P6 IC			✓			✓			✓
P6 IT	✓		✓			✓			✓

To further illustrate this finding, a participant's chosen strategies in the two tasks is presented. P17's chosen strategies in the Chord task in Static is shown in Figure 7-52 and in the Tangent task in Interactive is shown in Figure 7-53. P17's strategies in the two tasks vary. A summary of the use of the strategies in each task is presented.

Table 7-3 Strategies chosen by varying-instantiation participants in the Chord and Tangent tasks

Participant Task	Representation-specific						Re-representing		
	Graphic-algebraic	Graphic-numeric	Graph-wise	Numeric-algebraic	Numeric-trial	Point-wise	Symbolic	Textual	Visual
P7 DC			✓					✓	
P7 IT	✓		✓			✓		✓	✓
P8 IC			✓			✓			
P8 ST	✓		✓	✓		✓			
P9 DC			✓		✓				✓
P9 IT		✓	✓			✓			✓
P10 IC		✓	✓			✓			✓
P10 ST			✓			✓			✓
P11 SC			✓			✓		✓	✓
P11 DT	✓		✓			✓		✓	✓
P12 IC			✓						✓
P12 DT			✓			✓			✓
P13 DC			✓						
P13 ST		✓	✓			✓			
P14 SC			✓			✓			
P14 DT			✓			✓			✓
P15 DC			✓						
P15 IT		✓							✓
P16 IC			✓			✓			
P16 ST	✓	✓	✓						✓
P17 SC	✓		✓			✓			
P17 IT			✓			✓			✓
P18 SC	✓		✓			✓			✓
P18 DT		✓							

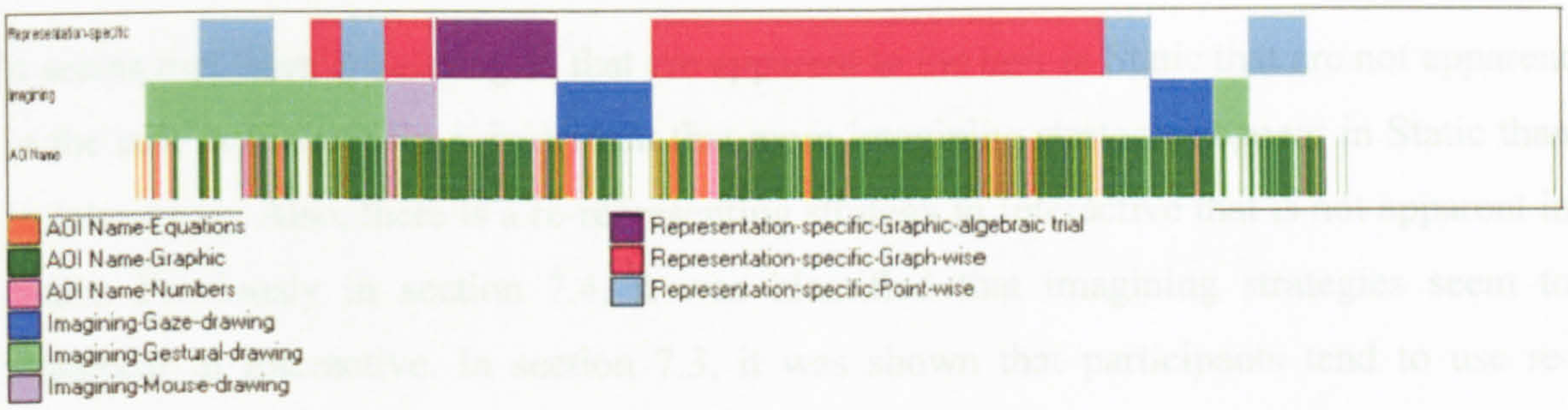


Figure 7-52 P17's chosen strategies in the Chord task in Static

P17 uses three representation-specific strategies in the Chord task in Static. He uses the point-wise strategy to relate the answer to the discrete points of the graph (i.e. the minimum and the maximum points). He attempts to use a graphic-algebraic strategy by looking at the graph and its corresponding equations. He typically uses the graph-wise strategy to describe what he perceives to be happening with the changes between one graphical representation in a particular figure, and to another graphical representation in another figure. He also uses three different kinds of imagining strategies. He uses the imagining strategies before and during describing a movement that a graphical representation makes. There is no apparent re-representing strategy identified.

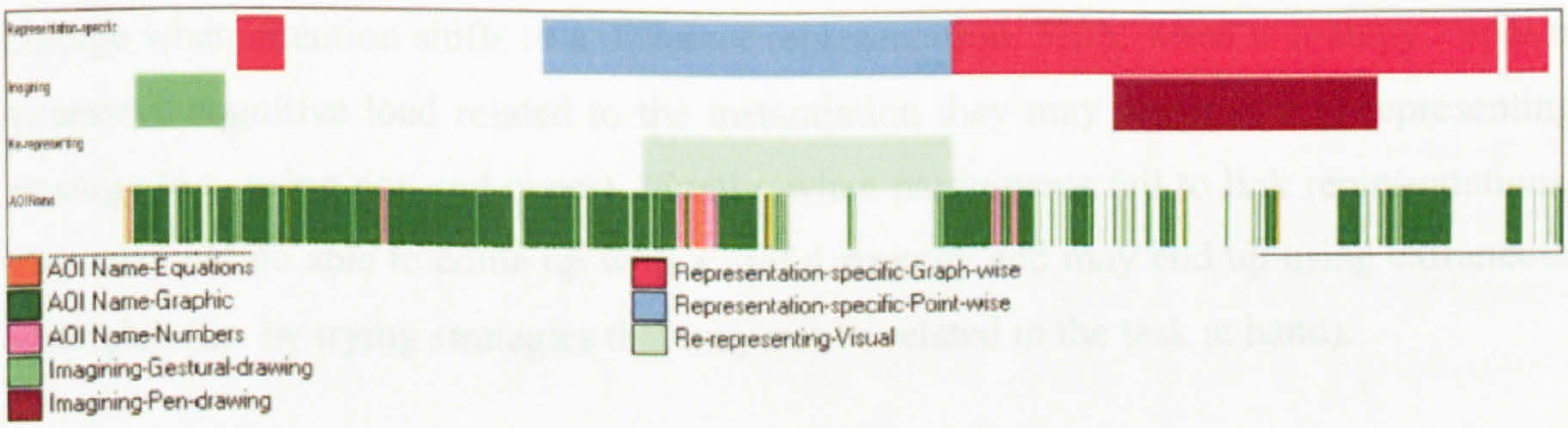


Figure 7-53 P17's chosen strategies in the Tangent task in Interactive

P17 uses two representation-specific strategies in the Tangent task in Interactive. He uses the strategy in the same way as in the Chord task. The occurrence of imagining strategies in the Tangent task is less than the imagining strategies apparent in the Chord task. There is an apparent use of re-representing strategy.

It seems that there are strategies that are apparent to the task in Static that are not apparent in the task in Interactive. It is evident that more imagining strategies appear in Static than in Interactive. Also, there is a re-representing strategy in Interactive that is not apparent in Static. Previously in section 7.4, it was identified that imagining strategies seem to disappear in Interactive. In section 7.3, it was shown that participants tend to use re-representing in Interactive.

7.8 CONCLUSION

This chapter has provided detailed evidence based on the integrated analysis of utterances, actions, gazes and sketches. The results support the findings of the quantitative data analysis. The following are speculations for some of the possible reasons for participants' changes in strategies. First, when participants experience difficulty in using a strategy related to the external mathematical representations (e.g. manipulation of equations) they change strategy. Second, when a participant experiences difficulty in visualising mental images, a graphic-related strategy may change to a different representation-specific strategy. Third, a participant may change strategy when a strategy is either difficult or easy to perform in relation to the interactivity feature of an instantiation. Fourth, a strategy related to a representation that has captured attention may change when attention shifts to a different representation. Fifth, when a strategy imposes excessive cognitive load related to the instantiation they may resort to a re-representing strategy (e.g. using pen and paper). Finally, when participants fail to link representations, they may not be able to come up with a useful strategy and may end up using extraneous strategies (i.e. by trying strategies that may not be related to the task at hand).

The examination of strategies with computer-based representations as evidenced by gazes, utterances, actions and sketches is providing richer explanations on how strategies change and how learners improve their strategies when multiple standard external representations with varying instantiations are involved.

The findings from Chapters 6 and 7 are reflected in the final chapter.

8 GENERAL DISCUSSION, LIMITATIONS, IMPLICATIONS, AND FUTURE RESEARCH

8.1 INTRODUCTION: FOCUS ON STRATEGIES AND INSTANTIATION

The literature (as discussed in chapter 2) provides evidence that learners using computer-based MERs experience difficulty related to standard external representations and to instantiations. Cognitive load theory has a potential to explain why learners have these particular difficulties (section 2.2.3). These difficulties appear to be due to intrinsic (i.e. inherent difficulties related to the external representation) or extraneous cognitive load (i.e. processing required due to the presentations of external representations) or can be a combination of both intrinsic and extraneous. Some examples of these difficulties are about the cognitive processing required in performing operations with standard external mathematical representations, viewing and manipulating computer-based representations, handling attentional resources when focusing on different elements of certain representations (see detailed lists in sections 2.2.1 and 2.2.2). Previous studies on computer-based representations suggest that the external representations given to learners are crucial to the decisions that learners take in choosing representations to tackle and in interpreting chosen representations (section 2.3.1). It is identified that different combinations of external representations in the design of a computer environment have an effect on learners' actions, conversations, (section 2.3.1) and self-constructed representations (section 2.3.3). The literature also gives an account of the capability of alternative instantiations such as animation, dynamic-linking, directly manipulable elements of graphs, for reducing difficulties in visualising graphical representations (section 2.3.2).

However, chapter 2 also notes that the claims in the literature about the effect on learning of varying external representations and varying instantiations are problematic because of inconsistent results from research into learning with computer-based representations. It is likely that the differences arise because the empirical evidence is not yet sufficiently precise; this may be due to the empirical approach taken in examining learning effects.

Ainsworth (2006) proposes an alternative way to examine learning effects of multiple representations, that the design of the external representations, the functions of the representations given and the task are all possible factors in these apparent conflicts. This thesis proposes that differences in instantiation, particularly the nature and degree of interactivity, are important; and the characterisation of strategies and the strategies learners use in choosing and interpreting representations are crucial to the successful completion of tasks. However, the literature on strategies and their association with specific representations is still sparse (sections 2.4.1 and 2.4.2). The research presented in this thesis takes strategy as its unit of analysis, which extends Aczel's (1998) research on examining learners' strategies. This thesis demonstrates that this approach can provide insight into the difficulties with the use of standard external mathematical representations to solve problems.

As described in chapter 0, this research included three types of mathematical representations (numeric, algebraic, graphic) in three instantiations (Static, Dynamic, Interactive). The choice of representations was based on standard practice, both paper- and computer-based, in mathematics and related disciplines. Mathematical representations were chosen because learners experience persistent difficulties with these representations. The choice of instantiations was based on an analysis of instantiations used in studies reported in the literature (e.g. Ardac and Akaygun, 2005; Lowe, 2003; Jones and Scaife, 2000; Otero, Rogers and du Boulay, 2001) and was informed by suggested guidelines for reducing cognitive load and assisting attention (Sweller and Chandler, 1994; Mayer and Moreno, 1998). Hence, key results based on existing literature about representations, interactivity, imagery, re-representation and strategies were considered in the identification of the research hypotheses and the design of the research.

This final chapter is organised as follows. First, there is discussion of the methodological contributions of this research, in terms of both data capture and analysis. The status of the research hypotheses in the light of the research is then examined. The subsequent section discusses the limitations of the methods and assesses their likely impact on the research findings. Then, some of the implications of this research for learning with computer-based MERs and for analysing computer interactions are suggested; and recommendations for future research are identified. The last section encapsulates the position of this research in examining strategies with varying computer-based representations.

8.2 INNOVATIONS IN METHODOLOGY: EVIDENCE BASED ON GAZES, ACTIONS, UTTERANCES AND SKETCHES

This research demonstrates a combination of techniques that afford a more precise view of what learners are doing when tackling representations in the context of learning using computer-based MERs.

8.2.1 CHARACTERISATION OF STRATEGIES WITH COMPUTER-BASED REPRESENTATIONS

The literature (section 2.4) provides justification for hypothesising that strategies with each standard external representation can be characterised at different levels of granularity. Furthermore, this research made additional observations about how learners associate strategies with representations, as reflected in the coding scheme in section 5.4.1. This research identified five classes of strategies used with computer-based MERs: representation-specific, imagining, re-representing, generic, and interface-supported. The evidence based on gazes, utterances, actions and sketches helped in identifying examples of the first three classes. The two latter classes of strategies were noted but not reported in this thesis because the former three classes of strategies offer the convenient way of examining strategies identified to address the hypotheses (as discussed in section 5.4).

The three classes of strategies of particular interest identified were representation-specific strategies (for viewing and manipulating particular representations); imagining strategies (in which participants appear to be engaged in mental visualisation); and re-representing strategies (how the learners present their inferences using representations not made available to them). The range of representation-specific strategies match the strategies identified by previous research (e.g. Keller and Hirsch, 1998; Ruthven, 1990; Even, 1998;

Villarreal, 2000; Senk and Thompson, 2006; Tabachneck et al., 1994); and also relates to the difficulties learners experience in tackling these specific representations (section 2.2.1).

Due to the availability of digital videos of gazes, actions, utterances and sketches, sub-classes of strategies within each class emerged from iterative refinement of characterisations of strategy choices (section 5.4.1). Examples of these are: graph-wise strategy (a representation-specific strategy characterised as descriptions of graphs' properties or behaviour), gaze-drawing strategy (an imagining strategy characterised as eye movements over the path of an invisible curve), visual re-representing strategy (a re-representing strategy characterised as a presentation of inferences in visual forms).

8.2.2 DIGITAL APPROACHES BASED ON GAZES, ACTIONS, UTTERANCES AND SKETCHES

The literature (chapter 2) offered indications that making multiple representations available encouraged learners to use other forms of representations when they had difficulties using their preferred representation. However traditional research techniques were inadequate to identify exactly which representations learners were using at any point in time. This research introduced a new combination of techniques including both video and eye tracking to address this deficiency.

Chapter 3 sections 2, 3 and 4 and the methodological challenges identified in the 2003 study (section 1.4) contributed to the hypothesis that analyses of strategies based on gazes, actions, utterances and sketches can provide factors contributing to strategy choice not possible with traditional observation techniques. In previous work (San Diego, 2003, as discussed in section 1.4), traditional video-based observation and analysis techniques were used. A variety of limitations were associated with that approach: it was typically not possible to tell exactly where on the screen the participant's attention was focused; the task of coding the data in detail was laborious; and transcript of talk, gesture and writing could not be easily combined simultaneously. These limitations are typical of research relying on video and traditional data capture, as described by Bigum and Gilding (1985) and reviewed in section 3.2.2.

Therefore, in this research, digital video was combined with state-of-the-art technologies such as eye-tracking and screen capture to provide a comprehensive record of participants' behaviours, including shifts in attention, utterances, and notes (section 3.4). The combination of eye-tracking, Tablet PC, screen capture, think-aloud, and video provided detailed data on what participants see, and what they do and say about what they see. These multiple video data with other forms of data such as event logs, AOI, Timelines, and transcripts (section 5.2.1, 5.2.2, and 5.3.4) were coordinated in cutting-edge analysis software (section 5.2). A set of analytical techniques was developed to analyse this integrated data, building on San Diego (2003), to identify learners' strategies with computer-based MERs (sections 5.3 and 5.4).

The analysis approach gave rise to a framework to deal with basic quantitative analysis. The preliminary results of analysing the number of participants with correct answers, time to task completion, number of participants using certain strategies, and AOI analysis were reported in chapter 6. The results in chapter 6 helped in organising the qualitative analysis of the differences in strategy use and strategy application. The findings of the qualitative analysis were presented in chapter 7. The main findings were related to differences in the use of strategy specific to external representations, variation of strategies for expressing inferences, the relationship of imagery and interactivity, the attention paid to representation, the other difficulties relating with mathematical representations and instantiations, and possible explanations for why and how changes in strategy occurred.

The particular combination of data collection techniques and integrated analysis provided unique evidence of visual imagery. For example, as illustrated in section 5.3.1, it was possible to see participants tracing graphs with their eyes to predict graph behaviour. Only the combination of eye-tracking and think-aloud could have so clearly revealed that they were trying to make sense of particular representations that appeared on the screen (indicated for example by the concentration of gaze on the displayed graph) and were predicting future behaviour (indicated by scan paths which moved from the displayed graph to 'the invisible curve': empty regions which would be occupied if the graph were manipulated according to the participant's predictions.)

Another example of the value of the methodological approach is in the combination of real-time records of think-aloud utterances, sketches, and gazes to provide insight into the moments at which participants feel the need to supplement the given representations with their own (section 5.3.2). Just as the relationship between utterances and gaze can provide evidence of what participants pay attention to (e.g. Eger, 2005; Hansen, 1991), relating what participants record on a Tablet PC to what they looked at on the screen can also indicate the subject of attention. Hence, even when participants do not speak, the combination of other data (i.e. gaze, gesture, writing, sketching) can reveal their cognitive activity.

Some participants abandon working on the screen and tend to think by sketching (Pirie, 1996). Recording exactly what the participants write in real time has provided a more comprehensive real-time capture of participants' activity; such as when they change or revisit what they had written or when they record something that is not presented to them. Direct capture of sketching on the Tablet PC afforded a clear, complete record which can be difficult to achieve using a video camera focused on pen and paper.

The use of both eye tracking and Tablet PC enabled more precise analysis of what participants were looking at and attending to than video and screen capture alone could do. For example, participants sketch as they describe their problem solving, such as: "this point on the graph..." Pointing gestures using the stylus were captured precisely and linked accurately to utterances and gazes. Hence the combination of eye tracking data and sketching data can provide clarification of vague descriptions and pronouns, so reactions such as "Whoa! That's interesting" (P15) can be interpreted accurately. Relating the signifier "That", with the position of gaze on an external representation on the screen, for example, can help identify which among the external representations "That" referred to.

In summary then, the innovative combination of digital research techniques developed in this study have addressed some of the practical and technical limitations of traditional video-based analyses of learners' computer interactions, as identified in the 2003 study and by Bigum and Gilding. It has also reproduced the advantages of video data identified in chapter 3. Some new technical and practical challenges do of course arise: these are discussed in section 8.4 below. But this research has demonstrated how integrated analysis of learners' gazes, actions, writing, sketches and utterances can better illuminate subtle cognitive strategies, and so increase the precision and scope of analysis. These innovations apply more generally to the study of work with computers, and are not restricted to the study of mathematical representations.

8.3 EXAMINING STRATEGIES WITH COMPUTER-BASED REPRESENTATIONS

Chapter 1 presented the main research question: **How do representations instantiated in different ways influence learners' cognitive processes?** In order to tackle the research question, the thesis articulated a set of hypotheses about learners' use of strategies, in the context of learning with mathematical computer-based representations. Each of these hypotheses is examined below in the light of the findings of the research. This thesis argues that understanding and explaining why difficulties with computer based representations occur is complex. The thesis proposed an alternative way of analysing learning effects from interactions with computer-based presentations by looking at strategies at different levels of characterisations. This section aims to highlight the potential of examining learners' strategies to precisely identify the reasons for the difficulties in learning with multiple representations. The attempts made in addressing the main aim of the thesis, to identify the effects on learners' strategies of varying computer-based representations, are outlined.

8.3.1 LEARNERS' CHOICE OF STRATEGIES WITH COMPUTER-BASED REPRESENTATIONS

It was hypothesised that learners' choice of strategies depends not just on the standard external representations given but also on the instantiation.

Sections 2.2.1 and 2.2.2 grouped difficulties according to standard external representations (Table 2-2) and to the type of instantiation (Table 2-3). Particular difficulties with standard external representations and with the type of instantiation were identified in section 7.6. In examining strategies, it was possible to relate the specific difficulty with standard external representation to a particular instantiation. An example given in section 7.6, about a difficulty related to visualising mathematical graphs (i.e. finding horizontal limits of a function) is similar to a difficulty in extracting numerical information from graphs (Dunham and Osborne, 1991). One type of instantiation suggests that the feature to generate a graph automatically (in Dynamic) could constrain the difficulty related to visualising graphs. This supports Elliot et al.'s (2000) finding that visualisation difficulty can be reduced by automatically produced graphs. However, quick and automatic generation of differently scaled graphs through manipulable elements of graphs by dragging (in Interactive) introduced problems in distinguishing, visualising and interpreting different graphs. This also confirms Weigand's (1991) finding. This suggests that different kinds of instantiations introduce different kinds of problems with one particular form of representation. In this research, participants were also observed having difficulty searching for comparable representations from a series of static images which sometimes lead to failure in visualising a graph's behaviour.

Task effect is an important factor to consider, as suggested by Cox (1996) and Ainsworth (2006). The original intention of this research was to offer three mathematical tasks of similar difficulty. Post-hoc analysis showed, however, that one of the tasks was easier than the other two (section 6.3). There is some evidence in this research that participants' choices of strategies vary between simple and complex tasks. For example in section 2.4.3, it was reported that learners have a tendency to perceive mathematical graphs as pictures (Weigand, 1991; Weigand and Weller 2001) when instantiation allows for quick generation of many graphs in a short time. In this research, participants could quickly generate mathematical graphs on the screen in two ways: through simple alpha-numeric input (in Dynamic) and by dragging manipulable elements of graphs about the screen using the mouse (in Interactive). Similar to previous research, it was found (section 7.2) that in the simple task participants seemed to view equations in a pictorial way, perhaps because the number of graphs learners need to process cognitively is overwhelming them.

At the same time, there was evidence that equations were viewed in a pictorial form (i.e. describing the equation not in terms of its mathematical meaning but rather in terms of whether the equation is long or short). This occurred in all three instantiations, but less so in Dynamic than Static or Interactive.

It was also found that the feature to generate graphs through alpha-numeric input, in Dynamic, may have encouraged participants to use algebraic-related strategies. Algebraic-related strategies in the simple task were only prominent in Dynamic and not in Static and Interactive (section 7.2).

Some researchers have reported that adding interactivity that controls the pace of a learning task reduces the time that people spend in completing a task (Evan and Gibbons, 2006). In section 7.4, in the complex task, participants spent more time using imagining strategies in Static than in Dynamic or Interactive. Yoon and Narayanan (2005) found that the use of an imagery strategy could increase time to task completion. The results of the research reported here to some extent corroborate this finding. However, it was found that there is no relation between instantiations and time spent to task completion (section 6.2). Nevertheless, the evidence presented in section 6.4 (based on AOI analysis) showed that there is a relation between instantiation and time spent on looking at each representation. Therefore, these findings can provide a strong speculation suggesting that the nature of the task, the degree of interactivity, the use of imagining strategies and the time to task completion are interrelated.

The evidence also showed that a choice of only graphic-related strategies did not elicit symbolic re-representing strategies; but elicited imagining strategies; in the simple task, a choice of algebraic-related strategies elicited symbolic re-representing but less imagining strategies. Furthermore, participants used imagining strategies less when they worked with instantiations that showed changes in the graphical representation. One of the possible reasons for this is their involvement in manipulating the graphical representations. Participants tend to use gaze-drawing with Static instantiations, whereas they tend *not* to use gaze-drawing with Interactive. More is said about this point in section 8.3.3 below.

8.3.2 MENTAL CONSTRUCTIONS OF IMAGES WITH GRAPHICAL REPRESENTATIONS

It was hypothesised that mental constructions of images with graphical representations vary between instantiations.

In section 7.3, evidence showed that in Dynamic and in Interactive, participants expressed inferences to compare different moments (or states) in a Dynamic or Interactive instantiation: they drew 'freeze frames' of the representation in order to compare it with other 'freeze frames'. This might be related to the progression of imagining strategies that participants choose, possibly experiencing difficulty storing changes in their working memory. There was some evidence that learners captured changes in gesture, for example following the movement of a point on a graph with a sweep of the hand (e.g. Figure 7-12 and 7-18). It might be that the more interactivity changes external representations, the more difficult it is for learners to store these external representations in their working memory. When the instantiation was Static, participants drew pictures that did not represent 'motion', rather their inferences were drawings of a final picture of graphs.

Section 7.4 provided some evidence of a 'progression' in using imagining strategies: it can be speculated that participants tend to use their eyes or mouse (or combination of both) in manipulating one representation; they then use their hands in comparing manipulations of two representations; and they use sketches or notes in comparing many representations. The change in imagining strategy during a task completion can be an indication that different imagining strategies apply for different kinds of representations. Whether a combination of imagining strategies is more effective than using one imagining strategy (c.f. Tabachneck, Koedinger and Nathan, 1994) needs further research. However, Cox (1996) pinpoints that externalising representations, among other things, can: reduce working-memory load; direct attention to certain parts of representations; aid information handling and retrieval; facilitate the inference of motion; facilitate shift of reasoning mode; and help express inferences. Cox suggests that externalisation can also be used to provide an indication of cognitive overload. The sequence of using imagining strategy from eye or mouse to gesture or pen may be related to increasing cognitive offloading. Gesturing to depict a picture of a moving image (Godin-Meadow et al., 2001) and visualising movements on paper (Cox, 1999) can reduce load in working memory. So, in tackling computer-based representations, before learners use gesture and pen to address their difficulty in visualising movement of graphical representations, they have the option to use the mouse or their mind (gazes and screen capture was used in this research to identify drawing happening).

The analysis of learners' strategies in Static highlighted another kind of difficulty which confirms Ardac and Akaygun's (2005) finding that learners have difficulty constructing moving images mentally. The evidence from sketches confirmed this (section 7.5.2). There is evidence that construction of images vary between instantiations (section 7.3). The choices of strategies relating to imagining appear to be different between Interactive and non-Interactive instantiations.

It is tempting to conclude that learners tend to reflect more when they use imagining strategies and less when they do not. It seems that when the representation and instantiation reduce extraneous cognitive load (Bodemer and Faust 2006), they may also reduce the need for imagining strategies. This is consistent with Larkin and Simon's suggestion that external representations play an analogous role to mental images, because the external (graphical) representation with the Interactive instantiation appears to be taking the place of imagining. Imagining strategies result in re-representation, which Cox

(1996, 1999) suggests promotes reflection. Hence reducing the need for imagining strategies may also reduce the opportunity for reflection.

8.3.3 ATTENTION PAID TO EACH STANDARD EXTERNAL REPRESENTATION

It was hypothesised that that attention paid to each standard external representation varies between instantiations.

As shown in section 6.4, participants' attention to external representations varied between instantiations within each task. Scanpaths showed that the majority of attention was given to the graph. Further qualitative analysis (section 7.4) showed that varying representational instantiations could draw the learners' attention to different aspects of representations. In Static, participants' attention focused on representations that they perceived would change and to the corresponding areas where a perceived movement of that change is going to happen. In Dynamic, learners must use alphanumeric input, drawing their attention to either numeric or algebraic representations. The sequence of manipulation associated with instantiation seems to influence the attention learners pay to representations. It was noted that when participants are required to enter numeric entries (such as in Dynamic) the split-attention effect can be reduced. It may be that it is the sequence of entering the number and looking at the effect on the graph that is beneficial. This is in line with the suggestions of Scaife and Rogers (1996) about dyna-linking: a manipulation of one representation which shows a change in another form of representation seems to influence learners' attention to representations.

In Interactive, the evidence also showed that when a graphical representation is being manipulated the participants tended to ignore other representations, such as numbers, presented with it. One of the possible reasons for this is their involvement in manipulating the graphical representations. This corroborates Van der Meij and de Jong's (2006) suggestion. In Dynamic, participants' attention tended to focus on entry and output; while in Interactive, the focus of attention is on what is being manipulated (e.g. a point being dragged about). This seems to be in line with the split-attention effect (Sweller and Chandler, 1991).

8.3.4 EXPRESSIONS OF INFERENCES

It was hypothesised that expression of inferences varies depending on the instantiation.

Participants made a variety of written inferences: algebraic, graphic, numeric, diagrammatic, textual, and combinations of those (section 7.3). In section 7.5, the evidence suggests that instantiation influences the way participants express their inferences. For example, in the simple task, if the instantiation is Static or Interactive, participants tend to make inferences that refer to graphs; however, in Dynamic, the inferences tend to be algebraic. In the other two tasks, if the instantiation is Dynamic or Interactive, participants tend to re-represent the 'movement' that a graphical representation makes; however, in Static, the inferences tend to represent the final perceived picture.

Consistent with the literature and as discussed in chapters 6 and 7, the ways participants expressed their inferences varied between representations. For example, participants typically made algebraic inferences from numeric and algebraic representations, and they typically made graphical inferences from graphical representations.

Learners seem to re-represent representations offered to them when they are having difficulty processing them (c.f. Cox, 1996), and they may prefer to work with the representations they construct. It can be speculated that varying instantiations influence learners' construction of representations because of the working memory load and also because access to representations may not be easy for certain types of representations.

8.3.5 HOW DO REPRESENTATIONS INSTANTIATED IN DIFFERENT WAYS INFLUENCE LEARNERS' COGNITIVE PROCESSES?

In answer to the research question, then, analysis of strategies with computer-based representations has shown that the type of instantiation affects learners' choice of representation, their propensity to construct mental images, their attention to different features of representations, and their expression of inferences. Understanding the relationship of performance or time to task completion on strategy remains a challenge for future research.

The strength of the analytical approach adopted is shown by a number of new hypotheses generated about the effect of different instantiations. For instance, it would be interesting to test in future research whether an Interactive instantiation makes a gaze-drawing strategy less likely, and focuses attention on what is manipulable; while Dynamic makes a pictorial interpretation of graphs less likely, and increases the likelihood of algebraic strategies, but also reduces the likelihood of imagining strategies.

The findings from examining the balance of strategies show that on a given task, learners use similar strategies to each other. Varying instantiations can influence learners' choice of strategies in relation to attention, expression of inferences, imagery and difficulty. This research found examples of participants using multiple strategies. There were situations when learners failed to link equations with corresponding graphs, and subsequently stuck to a single strategy based on one representation. When a strategy is tried and fails, or when participants experience difficulty in applying a strategy, then they change strategy. For example, if a participant using an algebraic strategy encountered difficulty in expressing inferences algebraically, the participant would dismiss that strategy and use a different one, such as a graph-related strategy. Instantiation can play a role here, for example, when participant P4 was in Dynamic, the alpha-numeric input, drew attention to values and prompted a change from a graph-wise strategy to a mixed graphic-numeric strategy. A further example is when making inferences about the movement of one element in a graphical representation, participants often begin with gaze or mouse strategies. Then, if they wanted to compare the movement of that element to the movement of another, they might add gesture. Then, when they wanted to compare more changing representations, they would shift to pen and paper.

With regard to interactivity and cognitive load, the relationship is probably not a simple one. Consider the following findings:

1) As discussed above, interactivity might reduce the need to re-represent, if it makes information evident that is required for the task and so reduces cognitive load.

2) On the other hand, in some cases interactivity appeared to increase the need to re-represent, because some of the information needed to solve the problem was not always visible. Section 8.1 discussed the difficulty learners experienced in capturing and storing changes in values, graphs, and equations.

3) High interactivity may produce additional cognitive load, related to the interaction itself.

One account might be that the impact of interactivity on cognitive load depends on the match between what information the interactivity makes evident and the particular informational requirements of the task. So, if the instantiation highlights key information in a way that directs attention appropriately, and the interaction requires the direct manipulation of key information (e.g. the input of values, dragging points on a graph), then cognitive load may be reduced.

Another account might be that the task, the representation and the nature of the interaction may require different encoding systems (e.g. visual and gestural), and that coordinating information between multiple cognitive systems has an impact on cognitive load. Hence, although simultaneous processing by multiple systems may be efficient when they can work largely independently, the requirement to coordinate between systems (e.g. when information from one encoding is required to fill in gaps in information in a different encoding) may increase load. This interpretation is speculative and requires further investigation. It does point the way to an investigation of strategies and representations from a neuroscience perspective.

8.4 CAVEATS: SOME LIMITATIONS OF THE RESEARCH

There were issues that could have influenced the findings of this research

- the constraints and potential bias associated with the study design
- the identification of strategies
- the materials used in the study
- the homogeneity of participants; and,
- validation of coding.

It is important to note the limits of the generalisability of the findings of this research in order to identify the aspects of the study that can be improved in future research.

8.4.1 CRITICAL EVALUATION OF THE STUDY

As is true of any laboratory study, a variety of factors could have introduced bias and influenced results. Therefore, caution must be applied in generalising the findings presented here, and further work is called for to rule out possible biases, for example through variations of setting, task and participants. Several of the more prominent challenges to generalisation are discussed here.

The study set-up could have affected the participants' task performance, e.g. the presence of the many types of equipment, the laboratory setting, the small payment given. There was a maximum length of time set for task completion, and this might have affected some participants' ability to complete the tasks fully. Participants' choices of strategies could be different were they given more time or less time to do the tasks. However, the analysis was not dependent on task performance, per se, and there was sufficient richness in participants' behaviour to sustain the analysis. Further work is required to assess the impact of the time limitation on strategy choice.

As has been discussed in chapters 6 and 7, this research does not rule out the possibility that some strategies may not have been captured. For example, one imagining strategy (identified in chapter 5), the mental-drawing strategy, was inferred from utterances alone. There may be other strategies which are not clearly manifest in the data collected (gazes, mouse actions, gesture, notes, utterances). Further, a different set of tasks, or a different configuration of representations or instantiations, might reveal more or different strategies. There is also the possibility that other researchers might identify different strategies from the same set of data. Hence, further studies are required before the strategies can be taken as general strategies. The key point here is that strategies have been demonstrated to be an effective and informative unit of analysis, one that sheds light on the discourse about learners' difficulties in using representations.

The age and mathematical background of participants were not homogenous. However, all participants did have mathematics A-levels, indicating that all had sufficient prerequisite knowledge to complete the tasks. Indeed, all were able to attempt the tasks, although not necessarily to complete them correctly. A rotational design was used to reduce the effect of prior knowledge. It remains a matter for further work to investigate the impact of prior knowledge on strategy and to consider other individual factors in the use of representations.

Because the tasks needed to be unfamiliar to participants, the tasks used in this study are not conventional questions that participants encounter in the classroom. They are similar to classroom questions in that they involve mathematical functions and require the sort of inferences expected in the solution of classroom problems. The representations involved are widely used in many subject domains.

The validation of coding was limited, as discussed in chapter 5. The coding scheme was validated only by one person, and only a sample of the data was dual-coded. The main issue with the validation is the requirement for the 'coder' to have sufficient knowledge of both mathematics and mathematics teaching, and sufficient familiarity with the data. Hence, full validation is prohibitively expensive. Therefore, although validation was attempted, it was limited because resources were not available to perform a full validation with an appropriately-experienced observer-rater. Nevertheless, the main limiting factor in this study was not the number of representations and instantiations being studied but the need to identify a broad range of strategies. Therefore, now that this study has succeeded in developing the data collection approach and analysis methodology, further studies in this area could focus on a small number of strategies across a number of representations and instantiations and obtain fuller validation.

Whilst the most common and widely applicable mathematical representations were chosen for this research, as described in chapter 4, variation of computer-based mathematical MERs might provide different results. Firstly, there are a huge number of other computer-based mathematical representations that are in use, such as spreadsheets, flowcharts, dynamic geometry, Latex, LOGO, calculators, numerical solvers, algebraic simplifiers, Venn diagrams, matrix manipulators, and the like. Secondly, there are also non-mathematical MERs such as toolbars, menus, WebPages, word processor documents, chemical structures, databases, architects' diagrams, engineering schemas, maps, video controls, calendars, etc. These non-mathematical MERs are part of the design of computer-based environments. Thirdly, the sequence of operation required due to the design of the computer-based MERs (e.g. inputting equations first before inputting numbers or inputting numbers before equations) may also impact on the strategy that learners may apply. Lastly, even within the three representations chosen there were alternative ways of implementing them (e.g. showing equations in the form (1/2)*x^2 rather than $\frac{1}{2}x^2$; or graphs that require scales to be entered; or the use of scientific notation, and so on) and alternative ways of implementing the instantiations (e.g. draggable sliders in Dynamic, or the display of four Static pictures per screen, or different ways of dragging in the Interactive instantiation).

8.4.2 CRITICAL EVALUATION OF THE DIGITAL APPROACH

The innovative combination of techniques for analysing and capturing learners' interaction with computer-based MERs has addressed some of the challenges identified in chapter 3. However, there are still some issues that should be addressed in future research.

A laboratory environment was needed primarily because of the eye-tracking equipment, a constraint which should be eliminated in the near future as the technology matures. Even then, the equipment had limitations with peripheral vision, spectacles, large movements of the head, rapid movements of the eyes, and unfocused gazes. On the other hand, unlike twenty years ago, the recording of screen, mouse, and keyboard activity can now easily be done in naturalistic settings without the need for controlled light and sound conditions. The recording of writing and sketching requires just a Tablet PC, although it feels less natural than using pen and paper. The recording of talk and gesture, meanwhile, requires a little more control over the environment, although it seems reasonable to expect that as less intrusive equipment such as wireless webcams and small wireless lapel microphones become more widespread, field research can grow to be the norm.

Capturing writing, utterances, actions, and gazes can produce an overwhelming amount of data. Knowing what to do with a large amount of data on complex interactions is a challenge. However, these data were collected in the same way as a researcher would conduct a think-aloud protocol analysis. The combination of techniques may also impinge on a given type of data collection. For example, combining think-aloud with eye-tracking may have an impact on timing, by introducing pauses for utterances. If one wants to use the eye-tracking data quantitatively, such as comparing the time participants take to look at a first target element, then pauses for utterances may bias results. It may be argued that participants might have dwelled more on what they are supposed to look at whilst describing what they are doing, than if they were just looking without describing.

The interface of INTERACT provides good support for iterative analysis models along the lines of Jacobs et al. (1999) and Powell et al. (2003). Selecting, setting up and operating data capture equipment still demands specialist skills; however, userfriendliness is steadily improving. Digital video cameras are now consumer items, and with the ubiquity of LCD monitors, screen flicker in video clips of participants is no longer problematic. Nevertheless, knowledge about video streams having different frame rates is needed because there can be a slight impact when synchronising the replay of multiple video streams. Moreover, there are choices to be made about codecs (programs for encoding and decoding video signals) that might not be available with some equipment combinations, and trade-offs between image quality and file size that can be tricky to judge. At this time, entry-level desktop computers are not yet adequate to manipulate video data effectively. Transporting, positioning, testing and looking after equipment still constitute a burden, although equipment is much less bulky than twenty years ago. Data storage media are smaller, more flexible, searchable, and less prone to degradation; however issues of labelling, backups and storage are still relevant. There are many costs, especially the eye-tracking equipment, the video analysis software and a computer with a high enough specification to handle video manipulation.

A major change from twenty years ago is participants' awareness of the issues of informed consent and potential misuse of data. The ease with which digital video can be distributed via the internet has changed participants' perception of vulnerability. Further, the use of audio and video recording limits the researcher's ability to anonymise data. Anonymising participants' faces still remains a challenge. There is technology to distort voices and blur faces, but it can remove crucial information, because intonation and facial expression can hold important cues.

A key improvement was in data coordination. Improved automation of time stamping for a variety of forms of data capture (e.g. software event logs, eye tracking logs, screen capture logs), and improvements in data handling software facilitated a combination of techniques. For example, the power to manipulate time stamps by entering an offset greatly reduced the time required to transcribe data and simplified synchronisation. Time spent coordinating and coding video recordings was much reduced with the help of analysis software tools that can simultaneously play all video streams from a specific moment.

Significant technical challenges remain. For example, time needs to be spent calibrating the eye-tracking equipment, and the equipment still has physical constraints. However, eye tracking equipment is improving steadily. Another example, is the challenge of identifying the start and end time of utterances. Automatic verbal transcription is not yet sufficiently robust, but it is reasonable to expect significant progress in this regard over the next few years. A Sound Analysis™ plug-in recently released by INTERACT may help address this challenge. It would also be ideal to have an eye-tracker that could be used for other devices such as Tablet PC or on mobile devices. A 'head-mounted' eye-tracker can be an alternative; however, this kind of device is perceived as intrusive and less accurate than an eye-tracker embedded on screen.

8.5 IMPLICATIONS FOR LEARNING WITH COMPUTER-BASED MERS AND FOR ANALYSING COMPUTER INTERACTIONS

On the basis of the findings of this research, possible implications are discussed for learning with computer-based representations, for teaching and learning with mathematical representations, for designing computer-based representations, and for analysing and capturing learners' interactions with computer-based representations.

8.5.1 TEACHING AND LEARNING WITH COMPUTER-BASED MERS

In relation to providing specific, well-founded guidelines for designing representations for specific learning tasks, this research extends understanding of the impact of representation and instantiation. It suggests that a suitable combination of representations can help learners to draw a variety of inferences, but only if they have reason to attend to each of the representations. Instantiation has a significant role, and it appears that interactivity bears a complex relation to cognitive load. The trade-off between the benefits of animating change and effect, and the costs of offering information that changes (and hence is not always visible) should be taken into account in making design decisions about instantiation. Further, the interaction itself carries its own overheads. Interactivity should be introduced when it relates to the task, and the nature of the interaction should be of a sort to reinforce and draw attention to key information.

Those who design or deploy MERs, such as teachers or educational software developers might find it helpful to be aware of these kinds of issues and trade-offs. They might also consider explicitly what kinds of strategies they are intending to foster, and what kinds of representations and instantiations might encourage or constrain these strategies. Rather than simply accepting that providing learners with multiple representations is *per se* a "good thing" (because being able to link multiple representations is perceived as a useful problem-solving skill and an indication of sophisticated thinking) or a "bad thing" (because they are known to confuse learners often), a more nuanced awareness of the role of computer-based MERs is desirable. Furthermore, bearing in mind the findings about cognitive load in this thesis, it might very well be appropriate for pen-and-paper to be available to students in many situations, even if the computer screen is intended to be the main focus of a particular task.

Further, teachers might be offered diagnostics that help them identify when representation use is not serving learning, or when particular behaviours indicate that students are not exploiting the MERs effectively. Teachers should be informed of the specific known pitfalls in using computer-based MERs. For example, to offset difficulties learners' experience in using zoom scales, teachers should guide students in interpreting graphs of different scales.

8.5.2 DIGITAL CAPTURE AND ANALYSIS OF COMPUTER INTERACTIONS

The digital approach to analysing learners' computer interactions developed in this thesis is applicable to a wide range of disciplines, such as engineering design, physics problem solving, computer programming, and so on. Further, it can be applied to other contexts, such as design, writing composition, and process control. However, people intending to use this technique would need to be aware that, like most video-based observation studies, the resultant data are rich and massive, and that therefore analysis needs to be carefully focused.

The digital approach offers great potential in sharing data, and in conducting meta-analysis across a number of studies. For example, automatic capture of screen, mouse and keyboard activity is now relatively uncomplicated, and it is now possible to share video on the internet (e.g. YouTube) and to search text captions of videos using certain search engines. Data repositories should become an established component in the research arsenal. However, this requires careful consideration and design in order that data can be shared, compared, and amalgamated meaningfully. Standards are required for data formats, transcription notations, analysis methods, and so on. For example, a standard way of presenting captions and the transcript of video needs to be agreed upon, otherwise, it would be impractical for researchers and readers alike to understand each data set presented differently. Also, video codec formats would ideally be free and easily accessible through the internet to avoid problems of compatibility playing videos in different computers.

Just as important as the technical challenges are the ethical ones concerning the sharing and distribution of learner data. There is a need to explicitly ask for participants' consent for using their video image or voice unaltered or altered.

8.6 RECOMMENDATIONS FOR FURTHER RESEARCH

This research required a substantial amount of time to be spent developing the data collection methodology and technical setup, designing the software, honing the analysis techniques, sifting the data, and identifying strategies. This research is suggestive rather than conclusive; each of the findings merits further study and validation. Systematic variation of tasks and participants would be appropriate in order to seek additional strategies, refine the findings, and tease out potentially confounding factors. The technology of data capture and analysis could be developed and improved further. The future of eye-tracking systems offers possibilities for designing systems for learning with computer-based representations.

8.6.1 IMPROVEMENT OF THIS RESEARCH

The intention of the 3-by-3 design was a systematic comparison across representations and instantiations. However, the performance of the participants brought the comparability of the tasks into question, and so it was not appropriate to make statistical comparisons across tasks. For example, it was observed that two of the tasks did not elicit any algebraic-related strategies whereas the other one did. Further, although the frequency with which participants used particular strategies was examined, the numbers were insufficient for statistical correlations between strategy and representation or instantiation to be established. One possible way to achieve this is to ideally come up with three similar tasks that have 'the same' level of difficulty and for which each version of the instantiations exactly offers the same manipulability, animation, and software operation. This may allow a comparison across, between and within tasks; and across, between and within instantiations, in a counter-balanced design similar to that used in this research. The results would then be amenable to statistical analysis. The effect on performance could be correlated with the strategy that learners choose using different representations. When results with a homogeneous group of participants are clear enough, the research could examine the effects of different levels of knowledge and experience on strategy choice. It is also worth considering using these research techniques to compare novices and experts, which might not only provide insight into the differences but could help to calibrate the analysis approach.

However, care should be taken in making generalisations based on quantitative eye-tracking data (e.g. comparing length and duration of fixations on specific elements). The sorts of metrics in the literature on eye-tracking studies may not apply to educational research. In particular, quantitative analysis of eye-tracking data collected during think-aloud is rather problematic, because (as discussed above) the introspection arguably alters the eye movement behaviour.

The data collected based on gazes, utterances, actions and sketches were very rich. The data can be subjected to further, multi-perspective analyses. For example, extensive qualitative analysis of scan paths, such as patterns in looking at different external representations, might have an influence on the choice of strategies. Another example would be examining whether differences in fixation patterns such as from-graph-to-numbers and from-numbers-to-graphs exist; and whether the differences can be associated with a difference in strategy choices. Furthermore, there was evidence in this research that the point-wise strategy disappeared in one of the tasks in Dynamic instantiation; it could be investigated whether there is a relationship between the time people spend looking at discrete points of a graph and the time people spend looking at numerical representations. This was not done in this research because there was no eye-tracking analysis software capable of achieving this. Recently, ClearView released a version capable of doing this.

The data from retrospective reporting has not been analysed fully. However, initial analysis hints that there is a mismatch in strategies reported between introspection and retrospection (c.f. Ericsson and Simon, 1984). The strategies that some of the participants reported seemed to exclude extraneous strategies; but extraneous strategies were identified during introspection. It seems that participants have a tendency to be selective in reporting only the strategies that they used successfully or the strategies that constituted major dead-ends. There are also strategies identified using the eye-tracking data that the participants did not report at all.

Some technical challenges remain, and new ones have arisen in capturing and analysing learners' computer interactions. For instance, there is a need to improve the data capture setup by automatically controlling the eye-tracker, digital video camcorder and the screen capture software to have a one-click synchronised recording. It would also be advantageous if infra-red cameras, used in eye-tracking devices, can accurately detect eye-movements in varying light conditions. Having this possibility, may allow for the data capture setup to be portable, rather than requiring participants to attend a laboratory study. It is also a challenge to use eye-tracking on small computing devices such as mobile phones, calculators and PDAs because of the small size of the screen. Eye-marks superimposed on representations on a small screen can conceal certain information (e.g. discrete points of a graph). Also, software tools for video analysis may be able to help in chunking data. The ability to share data easily on the internet and to do so in meaningful chunks may facilitate discussion among researchers in a way that helps to scrutinise, and potentially to validate, the coding scheme. In the long term, automated chunking, data sharing, and automated coding tools may enable an automated system for checking inter-rater reliability.

8.6.2 INTELLIGENT SYSTEM FOR CAPTURING LEARNING INTERACTION

Research with eye-tracking systems is moving forward. For example, eye-tracking is being incorporated in computers to control a computer's interface (see e.g. http://www.cogain.org/). The pedagogical benefit from eye-tracking systems is still in its infancy. Eye-tracking is now being explored with adaptive systems (e.g. Gütl et al., 2005), and eye-tracking is claimed to give indications of difficulty e.g. in reading text (Hansen, Hauland and Andersen, 2001). In the future, computing systems may include an intelligent interface that adapts to learners' needs by, for example, reducing the amount of external representations that are offered on the screen. This would be possible by utilising automatic analysis of AOI so that those elements that are being ignored, if important to the task, could give automatic feedback to learners; or if not important to the task, could be automatically hidden, to minimise the amount of information present on the screen.

8.6.3 VISUALISATIONS OF PEDAGOGY

Should an eye-tracking camera be embedded in personal computing, the educational world could benefit from the digital data it can offer. For example, it is illustrated in this research how eye-tracking systems can capture gaze data and user keyboard and mouse clicks logs. This data could be made available to tutors, teachers and lecturers in ways that provide indications of the kinds of interactions learners are having with representations. These visualisations can then be used to test the pedagogy of the design of computer-based representations.

8.6.4 EMERGING RESEARCH CHALLENGES AND HYPOTHESES

This research also raises some more focused questions. For example:

- Does a task that elicits a graphic-related strategy prompt more imagining and visual re-representing strategies than a task that elicits algebraic-related strategies? This seemed to be true of the three tasks in this study, but more tasks would need to be examined to investigate this more fully.

- Do people have movement or gesture preferences that constrain their interactions with (and inferences from) a direct-manipulation interface? For example, participants appeared to prefer to drag objects horizontally or vertically; if true, does this have an impact on their ability to make inferences that rely on a diagonal movement?

- How exact are people's 'invisible curves' and other gaze-drawing predictions? For example, in imagining a sequence of equally-spaced points, does their scan path show equally spaced fixations? In estimating distances, are their fixations proportional to the actual distances?

- Do people use a gesture strategy as a place holder in recalling the locations of a certain object that had been manipulated? Is this a form of spatial encoding?

- Is there a difference in learning outcomes for learners who use an imagining strategy from those who do not?

- What kinds of association are there between working memory load and learners' use of imagining strategies? What is the likelihood that learners' use of pen and paper during a computer-based activity is an indication of working memory overload?

- To what extent is learners' initial strategy choice governed by preference? Chapter 2 provides some evidence from the literature that learners have preferences for representations (e.g. Keller and Hirsch, 1998). Chapter 7 provided an example of a participant preferring an algebraic strategy but it is plausible that choices in general are influenced by the task, the given representation and the instantiations.

8.7 CONCLUDING REMARKS

The study presented here examined the strategy choice and use of 18 participants during their solution of three mathematics problems using three different representations and three different instantiations. A detailed analysis showed that both representation and instantiation have an impact on strategy choice. It identified differences in expression of inferences, construction of visual images, and attention to representations between different types of instantiation. One of the important findings of the research is that learners are less likely to use imagining strategies when representational instantiation is manipulable or moving. This research suggests that the impact of interactivity on cognitive load depends on the match between what information the interactivity makes evident with the particular information of the standard external representations. A possible way to decrease cognitive load is by introducing interactive features so that direct manipulation of key information (e.g. the input of values, dragging points on a graph) highlights key information of the external representation. The aim is to direct attention appropriate while the degree of interactivity gives learners control over the pace of a system to generate, for example, graphical representations. The results of this research may provide some explanation of how certain kinds of instantiations help or hinder learners' understanding of multiple representations.

The application of a novel combination of data capture techniques (eye tracking, think-aloud, screen capture, real-time sketching and writing, video) allowed detailed analysis of what learners say, do, see, and write while solving problems with multiple representations. A framework of strategies was derived, and innovative, integrated analysis methods were applied to tease out how interaction influences strategy and how strategy choice changes over the course of mathematical problem solving.

This research demonstrates how recent technologies can not only help to overcome some of the technical, practical and methodological challenges of using video to study learner-computer interactions, but also opens up new opportunities for capturing, coordinating and analysing video data relating to what learners say, do, see, and write when at a computer. Eye-tracking devices offer the chance to identify where learners are looking on the screen over time; while Tablet PCs offer the chance to capture writing and sketching in real-time. Software to coordinate multiple data streams can make analysis much more manageable and integrated. Transcripts can easily be augmented with gaze data, event data, gestures, extracts from participants' notes, and researchers' fieldnotes. Nevertheless, many challenges remain, and new ones have arisen. More work is needed to make these new research opportunities accessible.

This research provides some contributions into further developing the framework to examine strategies (Aczel, 1998) with multiple representations, particularly how strategies change and how learners improve their strategies when multiple external representations with varying instantiations are involved. It has begun to address the gap in the literature relating to the comparison of static, dynamic, and interactive representations. When learners are using multiple mathematical representations, their strategies are not mutually exclusive, as discussed in chapter 7. This is consistent with the literature, including Cox (1996), Borba and Confrey (1996), Villarreal (2000), Ainsworth (2006), and Goldin and Kaput (1996). Although there can be a strong association between a strategy and a given representation, the associations are neither unitary nor exclusive when representations are used in combination. Strategies may be carried from one representation to another as attention shifts. Strategies can be used in combination, and the combination of strategies may be prompted by the combination of representations and instantiations. But there is much more to understand, particularly the shifts of strategy that occur in diverse tasks with new and varied computer-based representations and instantiations. It remains a profound challenge to be able to design interfaces that adapt intelligently to learners' needs.

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APPENDIX A DETAILED PROCEDURES

PREPARATION: checklist of the equipments, materials, and procedures

- ☐ All files are ready
  - ☐ See information sheet
- ☐ All the equipments are in the lab
  - ☐ Tablet PC; Tablet PC stylus; Tablet PC cable charger
  - ☐ USB memory pen
  - ☐ USB keyboard; USB mouse; USB Adaptor
  - ☐ DVD
  - ☐ AAA batteries
  - ☐ A piece of rug
- ☐ All the materials are in the lab
  - ☐ Consent form
  - ☐ Index cards; field notes journal
  - ☐ Pencil/pen
  - ☐ Information sheet
  - ☐ Claim form
- ☐ All equipments are working and setup is complete
  - ☐ The video camera is recording properly
    - ☐ Enough space to record the session
  - ☐ The screen recording for tablet is working properly
    - ☐ Full screen input format
  - ☐ The eye-tracking camera is recording properly
    - ☐ Camera is auto-detecting

SETTING UP AND CALIBRATION

- ☐ Complete the information sheet
- ☐ Participants to play with the Tablet PC
  - ☐ Show pen button; eraser button; save button; page tab
  - ☐ Show the practice worksheet
    - ☐ Tell the participants that the worksheet is similar to the one that they are about to use.
    - ☐ Tell the participants the part of the worksheet
      - ☐ Where to record the information extracted and inferences; conjectures

- ☐ Ask the participant to practice writing on the worksheet using the:
  - ☐ Pen button ☐ eraser button ☐ save button ☐ navigation
- ☐ Ask them how they would prefer to have the Tablet PC positioned
  - ☐ Inclined OR ☐ flat on the table

PREPARATION FOR THE TASK

- ☐ Turn recording on
- ☐ Inform the participants briefly about the experiment and the research
  - ☐ ***READ:***

    This project is being done with the support of The Institute of Educational Technology, The Open University. This aims to investigate the processes you do in completing a problem on multiple mathematical representations such as equations, numbers and graphs.

    The study is using an eye-tracking device which detects where you are looking at. It is using an infra-red camera. The study will also record your activity and your voice using a video camcorder.

    Thank you for participating in this project. You are ensured that that the information we will collect from you will only be used for this project. Your personal details will be treated with utmost confidentiality and your opinion will be anonymised.
  - ☐ Ask the participant to sign the consent form
- ☐ Load the trial worksheet file
  - ☐ Ask the participant to read the task
  - ☐ Show the skin for the task
  - ☐ Ask them which they think will help them answer the task
  - ☐ Read the direction for the task
  - ☐ Ask for questions/clarifications
- ☐ Calibrate the participant
  - ☐ Right distance; right height; comfortable position
  - ☐ Remind the participant to avoid moving further than 2 feet from the screen
  - ☐ Ask again for questions/clarifications
  - ☐ Ask if they are ready to begin with the task

THE TASK

- ☐ Turn recording of all devices on (in order)
  - ☐ The video camera
  - ☐ The Tablet PC recording the total screen
  - ☐ The eye-tracking
- ☐ Turn recording of all devices on (in order)
  - ☐ STOP all recording ET, TAB, VIDEO
  - ☐ Save TAB worksheet and video, and ET

THE RETROSPECTIVE INTERVIEW

- ☐ Post interview
  - ☐ Show the video of the eye-tracking and the tablet
  - ☐ Ask the participant to talk through the task that has been completed

**AFTER EACH TASK*

- ☐ Whilst the participant is having a five minute break
  - ☐ Setup the next task
- ☐ Closing statement
  - ☐ Asking for participants if they can be contacted if there is a need for a follow up interview experiment at the OU
  - ☐ Reassure anonymity and confidentiality
  - ☐ Thank the participants

PRE-TASK INSTRUCTION

Calibration. In front of you is a monitor with an eye-tracking device that records what you are looking at on the screen. I need to calibrate your eyes for the accuracy of detection. 16 shrinking dots will appear one by one. I want you to look at the dots by avoiding swift head movements.

Tablet practice. Beside you is a Tablet PC where you are free to record any information that you wish to. You are only expected to use the pen button function and the eraser button. I have allotted a number of pages that you can use. You can move from one page to another by clicking on the page tab located at the bottom of the electronic worksheet.

The interface. Now I am going to show you the task and the corresponding interface for the task.

Question: Do you understand the task?
How are you going to answer the task without any software?
With the software, what element/representation do you think you will look at to help you answer the problem?

You have the freedom to use the Tablet PC to record whatever inference you have should you wish to. It is important that you say aloud what you are doing and what you are making sense of whilst doing the task. The task is followed by an interview where you will be asked to talk me through what you did.

Trial task

You are given a maximum of 3 minutes to complete that task. However, if you think you have answered the task, you can let me know, and we can move to the next part of the session.

You are to do the task using a Static version. You can view as many pictures as you wish from the 5 pairs of slides named slide 1a and 1b, 2a and 2b, and so on.

It is important that you say aloud what you are doing and what you are making sense of whilst doing the task.

Static Task instruction

You are given a maximum of 20 minutes to complete that task. However, if you think you have answered the task, you can let me know, and we can move to the next part of the session.

Tangential Region. You are to do the task using a Static version. You can view as many pictures as you wish from the 20 pairs of slides named slide 1a and 1b, 2a and 2b, and so on.

Root Variation: Same as above.

Chordal Midpoint: Same as above in tangential.

Dynamic Task Instructions

You are given a maximum of 20 minutes to complete that task. However, if you think you have answered the task, you can let me know, and we can move to the next part of the session.

You are to do the task using a Dynamic version. Please do not tick the Interactive option. You can change the scale of the Cartesian plane should you wish to but PLEAS DO NOT LEAVE IT BLANK.

Tangential Region. Using the software, you will need to enter a cubic function and a coordinate of a point not on the cubic then you can click animate to see the tangent through the point you have entered.

Root Variation. You will need to enter a cubic function, equation of vertical line, and the x-coordinate of chords endpoint. Then you can animate the chord and the vertical line.

Chordal Midpoint. You will need to enter a cubic function and the x-coordinate of point (a, 0), then click animate to see the rotated function through point (a, 0).

Interactive Task Instructions

You are given a maximum of 20 minutes to complete that task. However, if you think you have answered the task, you can let me know, and we can move to the next part of the session.

Tangential Region. You are to do the task using an Interactive version. Please do not untick the Interactive option. Using the software, there are points of the graphs that you can move using the mouse.

Chordal Midpoint: Same as above in tangential.

Root Variation: Same as above. For you to see the rotated function, you need to click on animate button.

You can also change the scale of the Cartesian plane should you wish to but PLEAS DO NOT LEAVE IT BLANK.

Retrospective report

I will show you the screen capture recording of what you did. This recording will also show what you were looking at during the task. Please talk me through what you did.
Questions asked: What happened here?

APPENDIX B TIME TO TASK COMPLETION

Time to task completion in Static

	Root			Chord			Tangent
P1	00:11:15:03		P1	00:28:20:07		P1	00:28:15:05
P2	00:14:25:19		P2	00:28:15:12		P2	00:20:44:10
P7	00:06:58:15		P11	00:18:20:18		P8	00:03:59:10
P9	00:04:16:15		P14	00:19:34:18		P10	00:12:48:17
P12	00:10:49:24		P17	00:13:47:12		P13	00:14:01:12
P15	00:08:23:07		P18	00:15:45:13		P16	00:28:07:04
Minimum	00:04:16:15			00:13:47:12			00:03:59:10
Maximum	00:14:25:19			00:28:20:07			00:28:15:05
Average	**00:09:22:22**			**00:20:04:18**			**00:17:10:06**

Time to task completion in Dynamic

	Root			Chord			Tangent
P3	00:11:12:15		P3	00:16:23:17		P3	00:19:22:17
P4	00:11:27:16		P4	00:21:00:10		P4	00:14:24:09
P8	00:22:25:09		P7	00:18:01:22		P11	00:28:02:03
P10	00:08:45:18		P9	00:14:25:06		P12	00:16:01:16
P16	00:27:12:23		P13	00:09:42:12		P14	00:24:46:11
P17	00:18:35:05		P15	00:16:44:02		P18	00:28:25:14
Minimum	00:08:45:18			00:09:42:12			00:14:24:09
Maximum	00:27:12:23			00:21:00:10			00:28:25:14
Average	**00:16:36:14**			**00:16:03:12**			**00:21:50:12**

Time to task completion in Interactive

	Root			Chord			Tangent
P5	00:06:29:15		P5	00:16:30:17		P5	00:19:20:11
P6	00:24:26:01		P6	00:22:15:22		P6	00:20:28:23
P11	00:06:49:02		P8	00:06:28:17		P7	00:07:44:22
P13	00:11:36:07		P10	00:16:50:17		P9	00:12:31:22
P14	00:11:40:11		P12	00:20:54:09		P15	00:08:27:03
P18	00:28:15:22		P16	00:22:32:24		P17	00:10:36:01
Minimum	00:06:29:15			00:06:28:17			00:07:44:22
Maximum	00:28:15:22			00:22:15:22			00:20:28:23
Average	**00:14:52:22**			**00:17:35:05**			**00:13:11:14**

APPENDIX C STRATEGIES CHOSEN BY TASK

Strategy used of participants

Participant/ Task	Representation-specific									Imagining					Re-representing		
	Algebraic-chunking	Algebraic-graphic	Algebraic-manipulation	Graphic-algebraic	Graphic-numeric	Graph-wise	Numeric-algebraic	Numeric-trial	Point-wise	Gaze	Gesture	Mental	Mouse	Pen	Symbolic	Textual	Visual
P1 SR	✓			✓		✓				✓			✓				
P2 SR					✓	✓		✓									
P7 SR					✓	✓		✓		✓			✓				
P9 SR					✓	✓				✓			✓				
P12 SR					✓	✓		✓								✓	
P15 SR			✓	✓	✓	✓	✓	✓									
P3 DR					✓	✓	✓	✓			✓					✓	
P4 DR				✓		✓			✓	✓	✓					✓	✓
P8 DR			✓	✓	✓		✓	✓			✓					✓	
P10 DR		✓	✓	✓		✓							✓				✓
P16 DR	✓	✓		✓	✓	✓	✓								✓		✓
P17 DR	✓		✓			✓	✓	✓									
P5 IR					✓	✓											
P6 IR	✓				✓	✓		✓			✓		✓		✓	✓	
P11 IR						✓		✓			✓					✓	
P18 IR	✓			✓	✓	✓	✓	✓							✓	✓	
P14 IR				✓		✓		✓					✓			✓	
P13 IR						✓				✓			✓				
P1 SC					✓	✓		✓	✓	✓			✓	✓			✓
P2 SC					✓	✓				✓		✓		✓			✓
P11 SC						✓			✓	✓	✓		✓	✓		✓	✓
P14 SC						✓			✓	✓			✓				
P17 SC				✓		✓			✓	✓	✓		✓				
P18 SC				✓		✓			✓	✓		✓		✓			✓
P3 DC						✓				✓	✓		✓			✓	
P15 DC						✓				✓	✓		✓	✓			
P7 DC						✓							✓			✓	
P9 DC						✓		✓					✓				✓
P13 DC						✓											
P4 DC					✓	✓				✓			✓	✓		✓	✓
P5 IC					✓	✓			✓				✓				✓
P6 IC						✓			✓		✓		✓	✓			✓
P8 IC						✓			✓				✓				

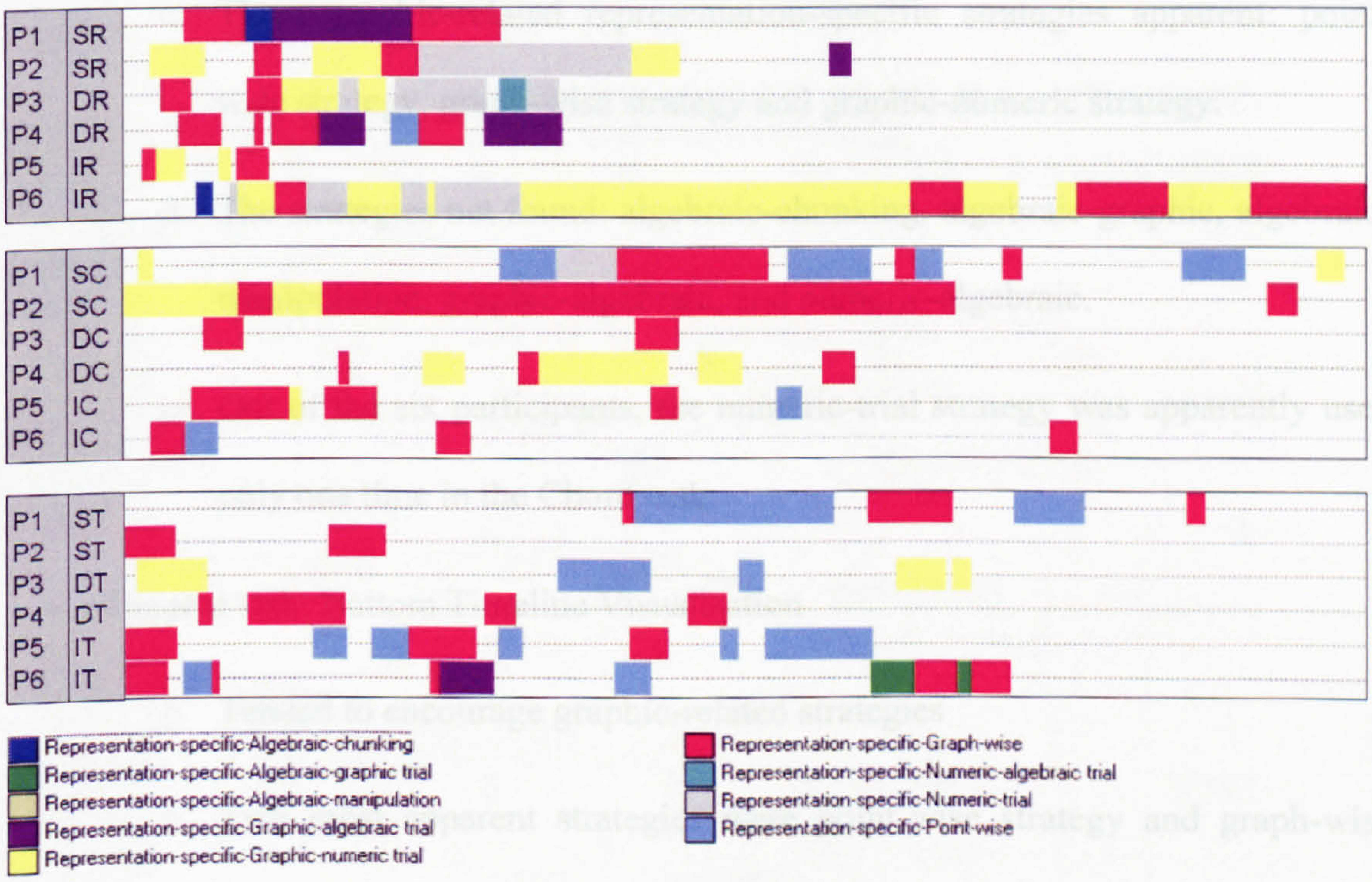
Participant/ Task	Representation-specific									Imagining					Re-representing		
	Algebraic-chunking	Algebraic-graphic	Algebraic-manipulation	Graphic-algebraic	Graphic-numeric	Graph-wise	Numeric-algebraic	Numeric-trial	Point-wise	Gaze	Gesture	Mental	Mouse	Pen	Symbolic	Textual	Visual
P10 IC					✓	✓			✓				✓	✓			✓
P12 IC						✓							✓	✓			✓
P16 IC						✓			✓				✓	✓			
P1 ST						✓			✓	✓		✓	✓				✓
P2 ST						✓				✓				✓			✓
P8 ST				✓		✓	✓		✓	✓							
P10 ST			✓			✓			✓	✓			✓				✓
P16 ST				✓	✓	✓				✓			✓	✓			✓
P13 ST					✓	✓			✓	✓	✓						
P3 DT					✓				✓	✓	✓		✓				✓
P4 DT						✓				✓	✓		✓			✓	✓
P11 DT				✓		✓			✓	✓			✓			✓	✓
P18 DT					✓									✓			
P14 DT						✓			✓	✓				✓			✓
P12 DT						✓			✓	✓			✓	✓			✓
P5 IT						✓			✓							✓	✓
P6 IT		✓		✓		✓			✓								✓
P7 IT				✓		✓			✓							✓	✓
P9 IT					✓	✓			✓		✓	✓					✓
P15 IT					✓								✓	✓			✓
P17 IT						✓			✓		✓			✓			✓

APPENDIX D ANALYSIS OF TIMELINES

ANALYSIS OF REPRESENTATION-SPECIFIC STRATEGIES BY TASK

Single-instantiation participants by tasks

There were six participants assigned to complete the three tasks with only one form of instantiation (P1 and P2 did the tasks in Static, P3 and P4 in Dynamic, P5 and P6 in Interactive). The strategies used for each of the three tasks (i.e. Root task, Chord task and Tangent task) are presented in Timeline Visualisations shown in Figure below.



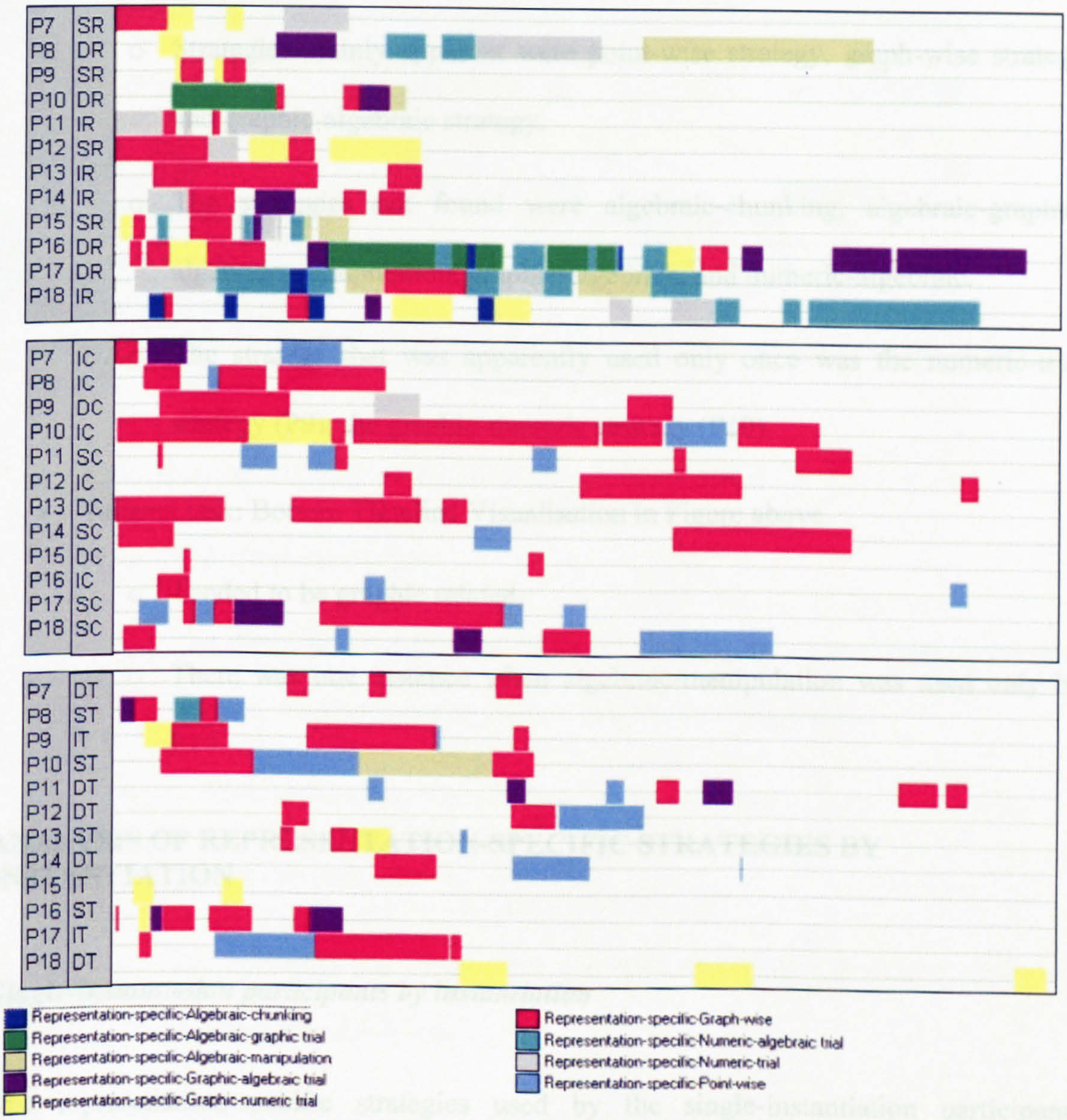
Representation-specific strategies for the single instantiation by task

- The Root task: top-most Timeline Visualisation in Figure below
    - This data suggests that the Root task tended to encourage the use of more or less all of the representation-specific strategies.

  - Only two representation-specific strategies did not appear among the strategies employed which were algebraic-graphic and algebraic-manipulation strategies.
  - Also, the strategies that appeared only once were numeric-algebraic strategy and point-wise strategy, of P3 and P4 respectively.
- The Chord task: Middle Timeline Visualisation
  - Tended to encourage graphic related strategies
  - Three graphic-related representation-specific strategies apparent: point-wise strategy, graph-wise strategy and graphic-numeric strategy.
  - The strategies not found: algebraic-chunking, algebraic-graphic, algebraic-manipulation, graphic-algebraic, and numeric-algebraic.
  - Out of the six participants, the numeric-trial strategy was apparently used only one time in the Chord task
- Tangent task: Bottom Timeline Visualisation
  - Tended to encourage graphic-related strategies
  - Two most apparent strategies were point-wise strategy and graph-wise strategy.
  - There were instances when graphic-numeric strategy was used by P3; and instances when P6 used algebraic-graphic strategy and graphic-algebraic strategy.

Varying-instantiation participants by tasks

Next, the strategies for twelve (12) participants who completed the three tasks with varying instantiations are analysed. The representation-specific strategies used for each task are shown in Figure below.



Representation-specific strategies for the varying instantiation collapsed by task

- Root task: Top-most Timeline Visualisation in Figure above

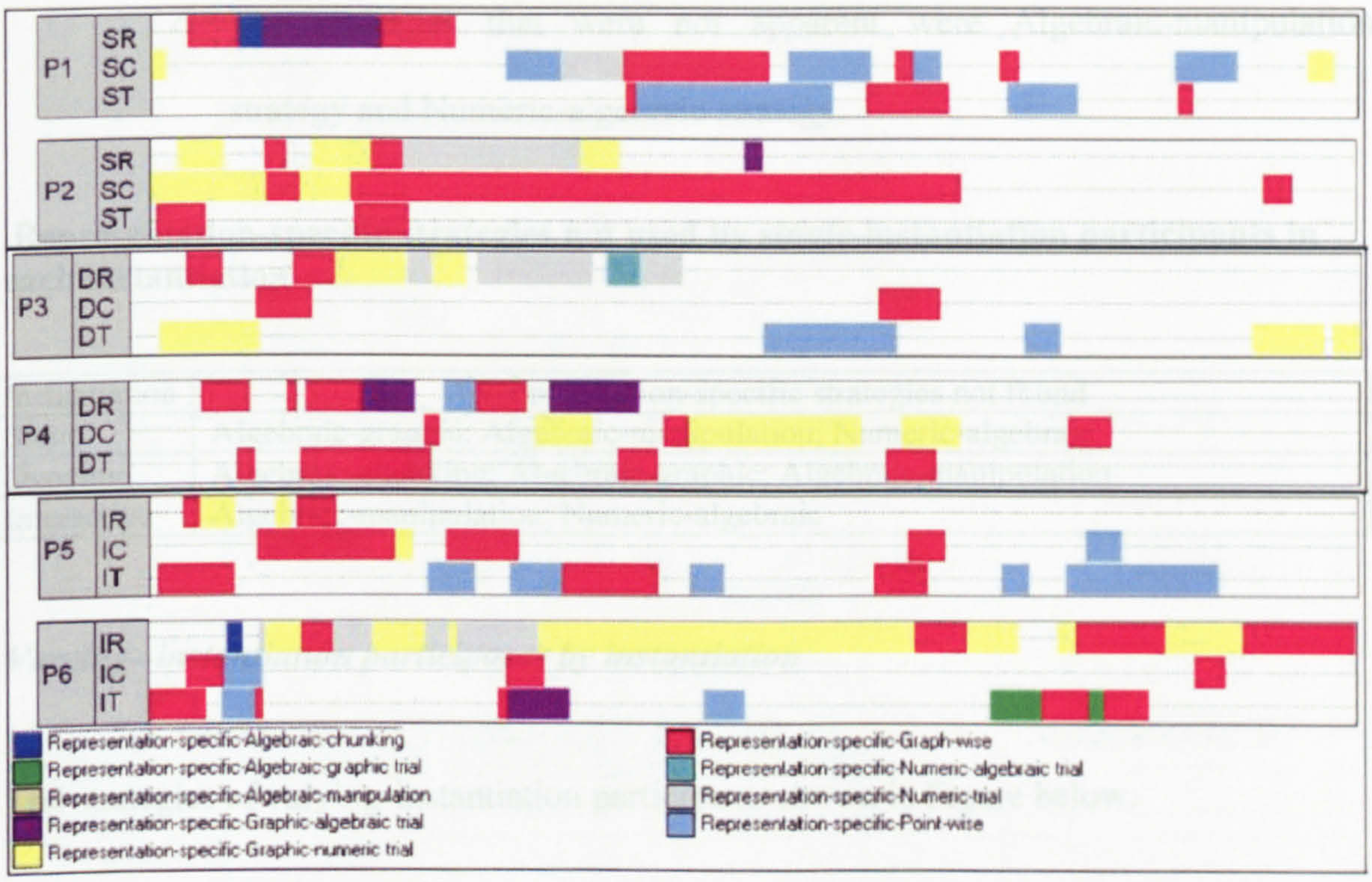
- Tended to encourage the use of more or less all of the representation-specific strategies.
- Only the point-wise strategy was not apparent.

- Chord task: Middle Timeline Visualisation in Figure above
  - Tended to be graphic related
  - Strategies mainly apparent were point-wise strategy, graph-wise strategy and graphic-algebraic strategy.
  - The strategies not found were algebraic-chunking, algebraic-graphic, algebraic-manipulation, graphic-algebraic, and numeric-algebraic.
  - The strategy that was apparently used only once was the numeric-trial strategy (P9); the graphic-numeric strategy (P10).
- Tangent task: Bottom Timeline Visualisation in Figure above
  - Tended to be graphic related.
  - There was one instance when algebraic-manipulation was used only by P10.

ANALYSIS OF REPRESENTATION-SPECIFIC STRATEGIES BY INSTANTIATION

Single-instantiation participants by instantiation

The representation-specific strategies used by the single-instantiation participants collapsed by instantiation are first presented below in the Timeline Visualisation in Figure below.



Representation-specific strategies for the single-instantiation collapsed by instantiation

- Static: Top-most Timeline Visualisation of Figure above
  - Vary from each other
  - The representation-specific strategies that did not appear were Algebraic-graphic, Algebraic-manipulation, and Numeric-algebraic.
- Dynamic: Middle Timeline Visualisation in Figure above
  - Vary from each other
  - The strategies not found were algebraic-chunking, algebraic-graphic and algebraic-manipulation.
- Interactive: Bottom Timeline Visualisation in Figure above
  - Vary from each other.

- The strategies that were not apparent were Algebraic-manipulation strategy and Numeric-algebraic strategy.

Representation-specific strategies not used by single-instantiation participants in each instantiation

Instantiation	Representation-specific strategies not found
Static	Algebraic-graphic; Algebraic-manipulation; Numeric-algebraic
Dynamic	Algebraic-chunking; Algebraic-graphic; Algebraic-manipulation
Interactive	Algebraic-manipulation; Numeric-algebraic

Varying–instantiation participants by instantiation

The strategies of varying-instantiation participants shown in Figure below.

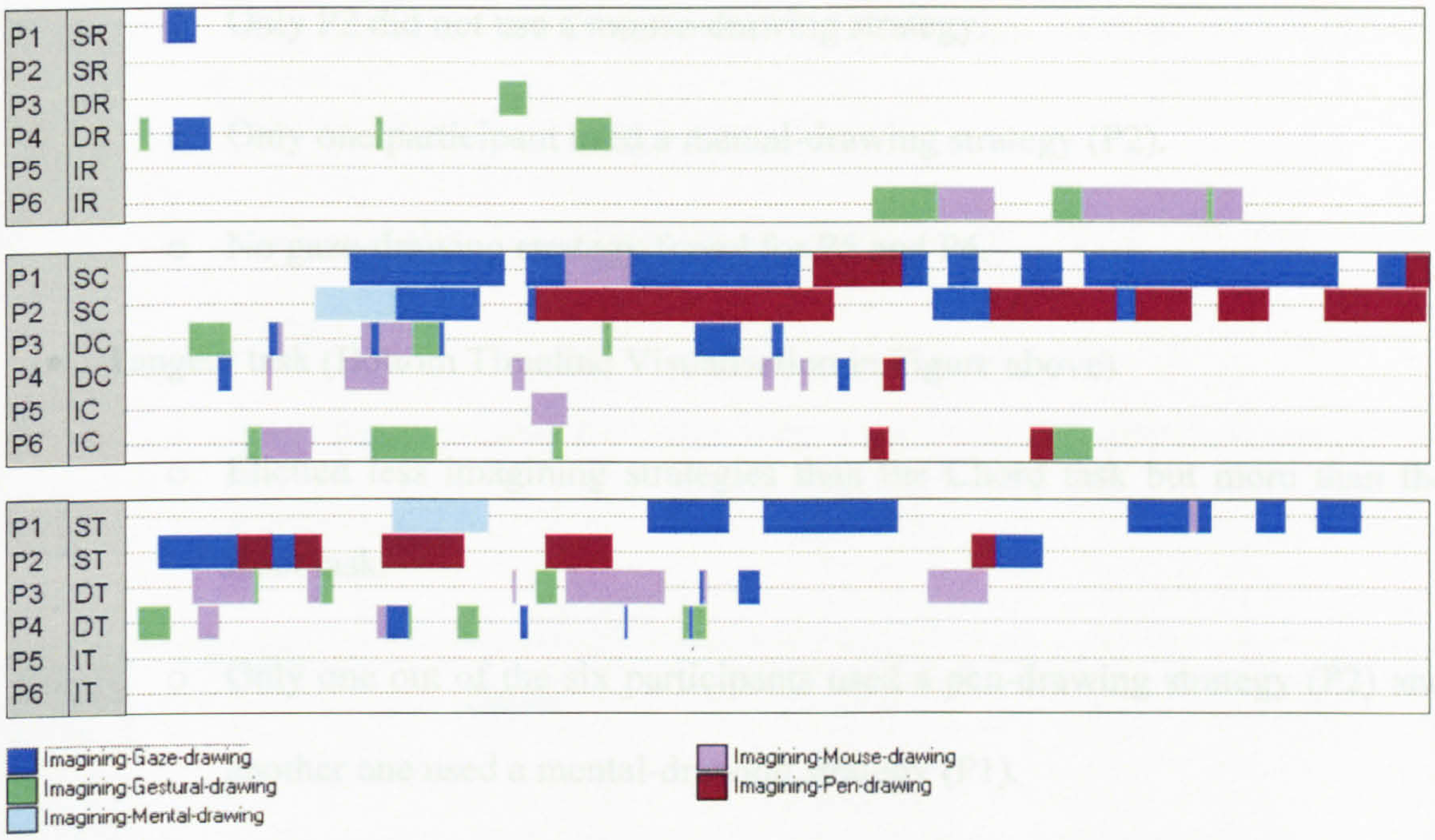
- Static: Top-most Timeline Visualisation in Figure below
  - Tended to encourage the use of more or less all of the representation-specific strategies
  - Algebraic-chunking strategy and algebraic-graphic strategy were not apparent.
- Dynamic: Middle Timeline Visualisation in Figure below
  - All of the representation-specific strategies were apparent.
  - Algebraic-graphic strategies were used by two participants (P10 and P16) and the algebraic-manipulation strategy by three participants (P8, P10 and P17).
- Interactive: Bottom Timeline Visualisation
  - Only Algebraic-chunking strategy and algebraic- algebraic-manipulation were not apparent.

Representation-specific strategies for the varying instantiation by instantiation

ANALYSIS OF IMAGINING STRATEGIES BY TASK

Single-instantiation participants by tasks

The imagining strategies used for each of the three tasks are presented in Timeline Visualisations shown in Figure below for single-instantiation participants.



Imagining strategies for the single-instantiation collapsed by task

- Root task (Top-most Timeline Visualisation in Figure above).
    - Did not elicit as many imagining strategies as the other two tasks.
    - No apparent pen-drawing strategy and mental-drawing strategy
    - Two participants used a gaze-drawing strategy (P1 and P4).
    - Two participants used a mouse-drawing strategy (P1 and P6).
    - No gaze-drawing strategy found for P5 and P6.
- Chord task (Middle Timeline Visualisation in Figure above)
    - Elicited more imagining strategies than the other two tasks
    - Most of the participants used a pen-drawing strategy and a mouse-drawing strategy.
    - Two of the six participants did not use the pen-drawing strategy (P3 and P5)

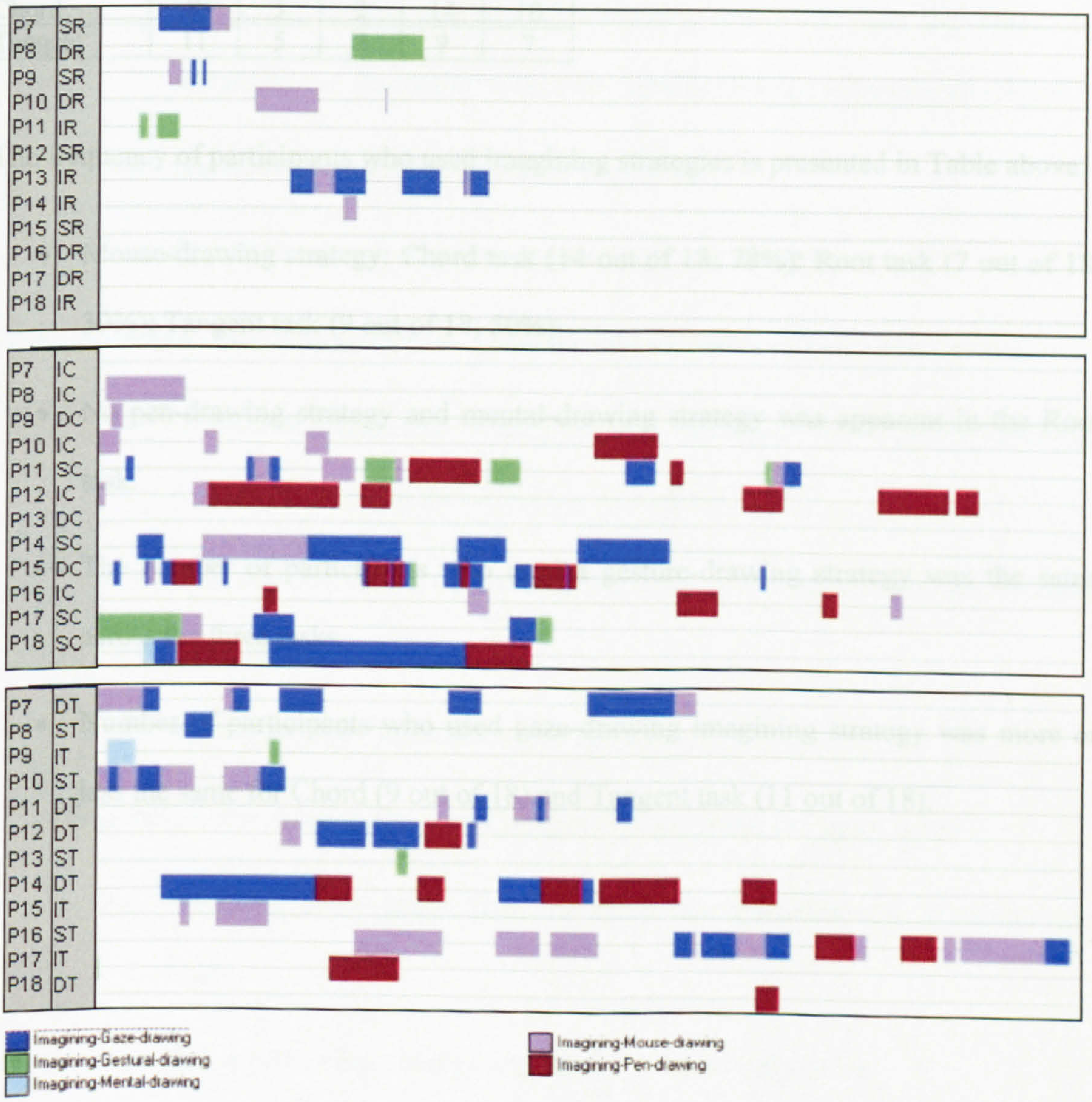
  - Only P2 did not use a mouse-drawing strategy.
  - Only one participant used a mental-drawing strategy (P2).
  - No gaze-drawing strategy found for P5 and P6.
- Tangent task (Bottom Timeline Visualisation in Figure above)
  - Elicited less imagining strategies than the Chord task but more than the Root task.
  - Only one out of the six participants used a pen-drawing strategy (P2) and another one used a mental-drawing strategy (P1).
  - No imagining strategy was apparent for P5 and P6.

Varying-instantiation participants by tasks

The imagining strategies used by varying-instantiation participants used for each task are shown in Figure below.

- Root task
  - Elicited only a few imagining strategies.
  - The pen-drawing strategy and mental-drawing strategy were not apparent.
- Tangent and the Chord task
  - Elicited similar imagining strategies.
  - More imagining strategies were apparent in the two tasks than in the Root task.

- The strategies mostly used were pen-drawing strategy and a mouse-drawing strategy.
- More pen-drawing strategies occurred in the Chord task than in the Tangent task.
- More gaze drawing strategies were seen in the Tangent task than in the Chord task.



Imagining strategies for the varying-instantiation collapsed by task

Comparison of strategies between single-instantiation and varying-instantiation participants by tasks

Number of participants using imagining strategies by task

Task	Frequency of participants who used the strategy below				
	Gaze-drawing	Gesture-drawing	Mental-drawing	Mouse-drawing	Pen-drawing
Root	5	5	0	7	0
Chord	9	5	2	14	10
Tangent	11	5	2	9	7

The frequency of participants who used imagining strategies is presented in Table above.

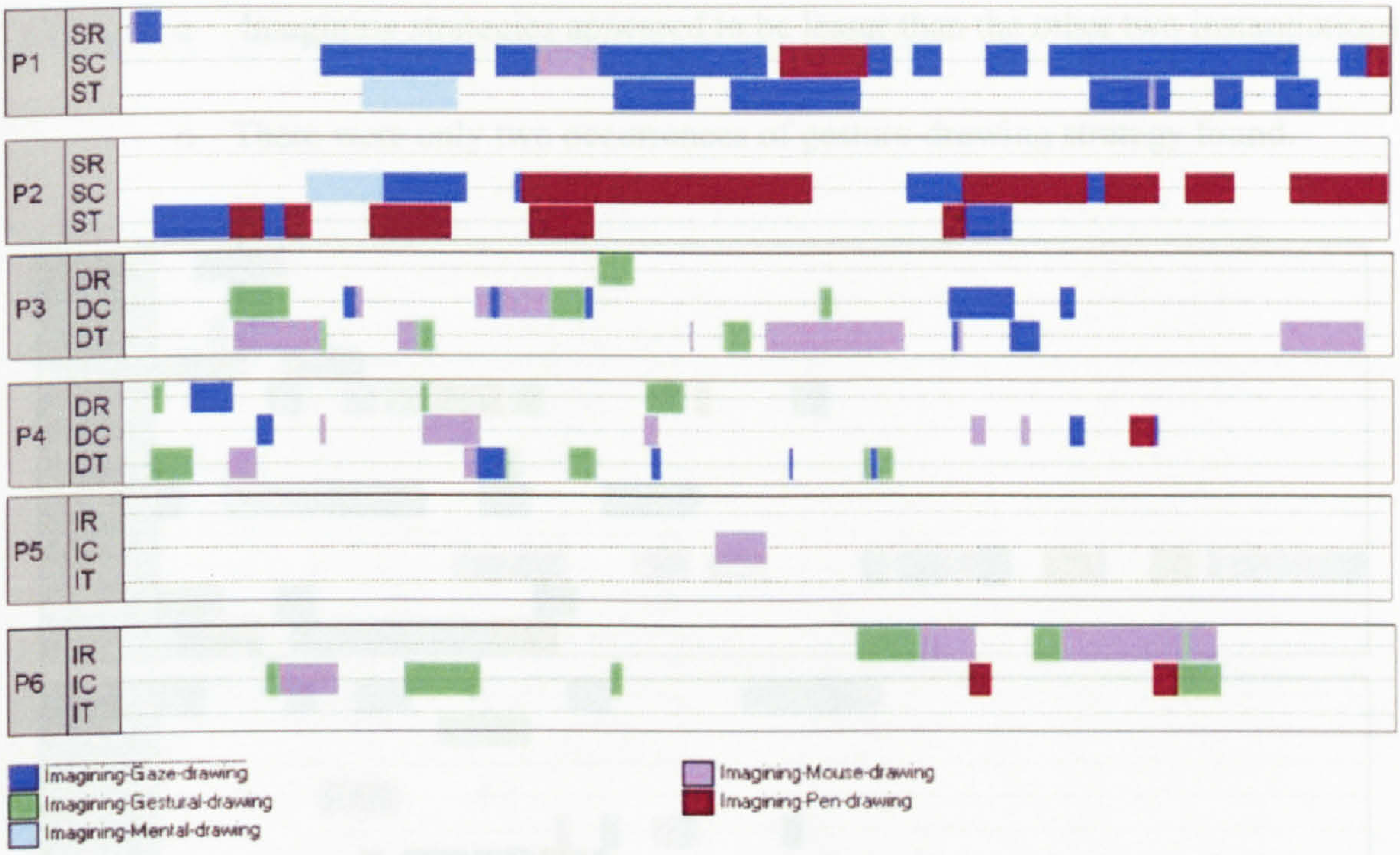
- Mouse-drawing strategy: Chord task (14 out of 18; 78%); Root task (7 out of 18; 39%); Tangent task (9 out of 18; 50%).
- No pen-drawing strategy and mental-drawing strategy was apparent in the Root task.
- The number of participants who used a gesture-drawing strategy was the same across the three tasks.
- Number of participants who used gaze-drawing imagining strategy was more or less the same for Chord (9 out of 18) and Tangent task (11 out of 18).

ANALYSIS OF IMAGINING STRATEGIES BY INSTANTIATION

Single-instantiation participants by instantiations

The imagining strategies used single-instantiation participants are presented in the Timeline Visualisation in Figure below.

- Static instantiation (Top-most Timeline Visualisation in Figure below)
  - Elicited a longer duration of imagining strategies than the other two instantiations. Particular to the pen-drawing and gaze-drawing strategies.
  - Gesture-drawing strategy was not apparent.
  - Only P1 used a mouse-drawing strategy; one occurrence.
- Dynamic instantiation (Middle Timeline Visualisation in Figure below)
  - Dominant imagining strategies that appeared were gesture-drawing, mouse-drawing and gaze-drawing strategies.
  - Only one occurrence of pen-drawing strategy was apparent in the Dynamic instantiation.
  - No mental-drawing strategy was apparent.
- Interactive instantiation (Bottom Timeline Visualisation in Figure below)
  - Appeared to be lesser than the other two instantiations, Dynamic and Static.
  - The pen-drawing strategy apparently occurred only twice.

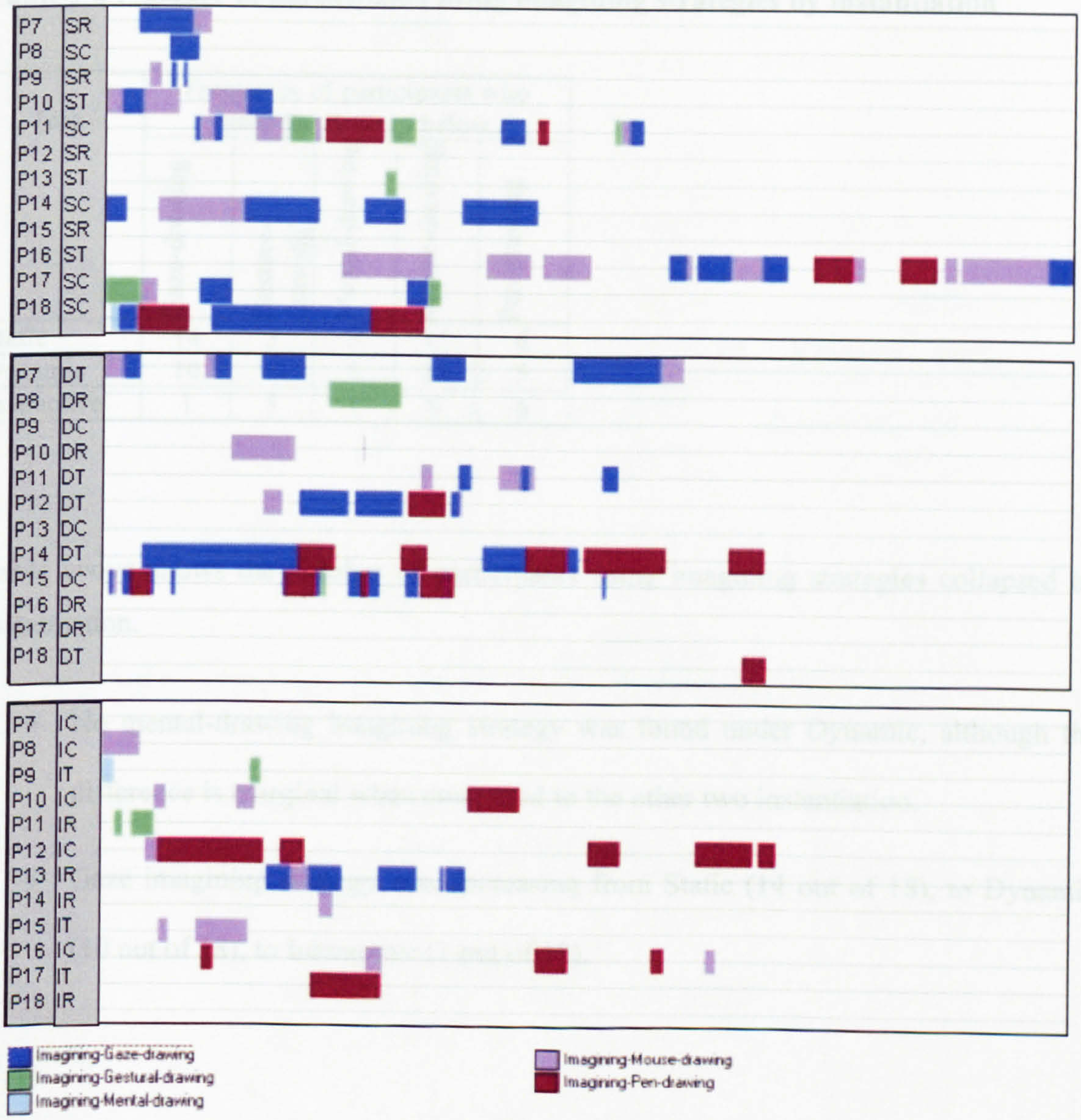


Imagining strategies for the single-instantiation collapsed by instantiation

Varying-instantiation participants by instantiation

- Static instantiation (Top-most Timeline Visualisation in Figure below)
  - Used longer duration of imagining strategies than the other two instantiations.
  - All imagining strategies were apparent in Static.
  - Gaze drawing-strategies were apparent for most of the participants.
- Dynamic instantiation (Middle Timeline Visualisation in Figure below)
  - Elicited lesser imagining strategies than the Static.
  - Only mental-drawing strategy was not apparent
  - There were only two occurrences of gesture-drawing strategy found.
- Interactive instantiation (Bottom Timeline Visualisation in Figure below)

- Imagining strategies appeared to be lesser than the other two instantiations
- There were only two occurrences of gesture-drawing strategy found.



Imagining strategies for the single- instantiation collapsed by instantiation

Comparison of strategies between single-instantiation and varying-instantiation participants

Table 9-1 Number of participants using imagining strategies by instantiation

Task	Frequency of participants who used the strategy below				
	Gaze-drawing	Gesture-drawing	Mental-drawing	Mouse-drawing	Pen-drawing
Static	14	3	3	11	6
Dynamic	10	7	0	10	5
Interactive	1	5	1	9	6

Table above shows the number of participants using imagining strategies collapsed by instantiation.

- No mental-drawing imagining strategy was found under Dynamic, although the difference is marginal when compared to the other two instantiation.
- Gaze imagining strategy was decreasing from Static (14 out of 18), to Dynamic (10 out of 18), to Interactive (1 out of 18).

ANALYSIS OF RE-REPRESENTING STRATEGIES BY TASKS

Re-representing strategies collapsed by task

Comparing single-instantiation and varying instantiation participants

Figure above shows the re-representing strategies collapsed by task.

- Re-representing strategies found for single-instantiation participants were also found for varying-instantiation participants.
- The Root task elicited more symbolic and textual re-representing strategies that the other two tasks.
- There was no symbolic re-representing strategy found under the Chord and Tangent tasks. The Tangent task elicited the more visual re-representing strategies than the other two tasks.

Comparing single-instantiation and varying instantiation participants

Figure below present the re-representing strategies collapsed by instantiation.

- The Static instantiation did not elicit many re-representing strategies than the other two instantiations.
- Minimal textual re-representing strategy used in Static.
- No symbolic re-representing strategy found in Static. Difference of symbolic strategy between instantiations is marginal

ANALYSIS OF RE-REPRESENTING STRATEGIES BY INSTANTIATION

Re-representing strategies collapsed by instantiation

Root

Instantiation	*Representation-specific*	*Imagining*	*Re-representing*
Static	Algebraic-graphic Point-wise	Gesture-drawing Mental-drawing Pen-drawing	Symbolic Visual
Dynamic	None	Mental-drawing Pen-drawing	None
Interactive	Algebraic-graphic Algebraic-manipulation Point-wise	Mental-drawing Pen-drawing	Visual

Chord

Instantiation	*Representation-specific*	*Imagining*	*Re-representing*
Static	Algebraic-chunking Algebraic-graphic Algebraic-manipulation Numeric-algebraic	None	Symbolic
Dynamic	Algebraic-chunking Algebraic-graphic Algebraic-manipulation Graphic-algebraic Numeric-algebraic Point-wise	Mental-drawing	Symbolic
Interactive	Algebraic-chunking Algebraic-graphic Algebraic-manipulation Numeric-algebraic Numeric-trial	Gaze-drawing Mental-drawing	Symbolic

Tangent

Instantiation	*Representation-specific*	*Imagining*	*Re-representing*
Static	Algebraic-chunking Algebraic-graphic Numeric-trial	none	Symbolic Textual
Dynamic	Algebraic-chunking Algebraic-graphic Algebraic-manipulation Numeric-algebraic Numeric-trial	Mental-drawing	Symbolic
Interactive	Algebraic-chunking Algebraic-manipulation Numeric-algebraic Numeric-trial	Gaze-drawing	Symbolic

APPENDIX E VARIATION IN INFERENCES

Written inferences for the Root task in Static

P1: correct: graphic

Each cubic function has 3 roots in which $y = 0$. If, for example, the 3 roots of $f(x)$ have x coordinates lower than the x coordinate of the point of rotation, then the x coordinates of the 3 roots of $g(x)$ will be the same distance from the point of rotation as those for $f(x)$, but in the opposite direction.

P7: correct: graphic

Number of roots is the same for each pair of curves.

Spacing between the roots stays the same — but [illegible] ..!

P12: correct: graphic

The roots of the new function $g(x)$ are equivalent distances away from the rotation coord. along the x-axis, i.e the closest root to the rotation coord in $f(x)$ is also the closest (+ same distance!) in $g(x)$.

P2: correct: graphic

The solutions of $g(x)$ are reflections around $x = a$ of the function $f(x)$.

P9: correct: graphic

The roots of the two functions are of equal distance from the point of rotation.

P15: correct: algebraic

$y = 2a - x$

where x = roots of $f(x)$

y = roots of $g(x)$

a = point rotated about.

Written inferences for the Root task in Dynamic

P3: correct: algebraic/numeric

roots are the inverse sign of original roots plus double the x value of the centre of rotation.
So if original roots are (-2, -2, -2) & centre of rotation is (3, 0) then new roots are: (8, 8, 8)

P4: not that correct: graphic/algebraic

If cubic function rotated about the point of inflexion at 180°, the roots are the same.

If the cubic function is rotated about a point (a,b) then the roots of the cubic is shifted by an

P8: correct: algebraic

Root of $g(x)$ is negative of the root of $f(x)$

CONJECTURE 2

Root of $g(x)$ is root of $f(x)$ $+ 2(a - \text{root of } f(x))$

LET $R_2(g(x))$ IS $R_2(f(x))$ after it's been rotated. (R = root, 2 = integer).

CONJECTURE 3 (CONJECTURE 2 REWRITTEN)

$R_2(g(x)) = R_2(f(x)) + 2(a - R_2(f(x)))$

P10: correct: algebraic

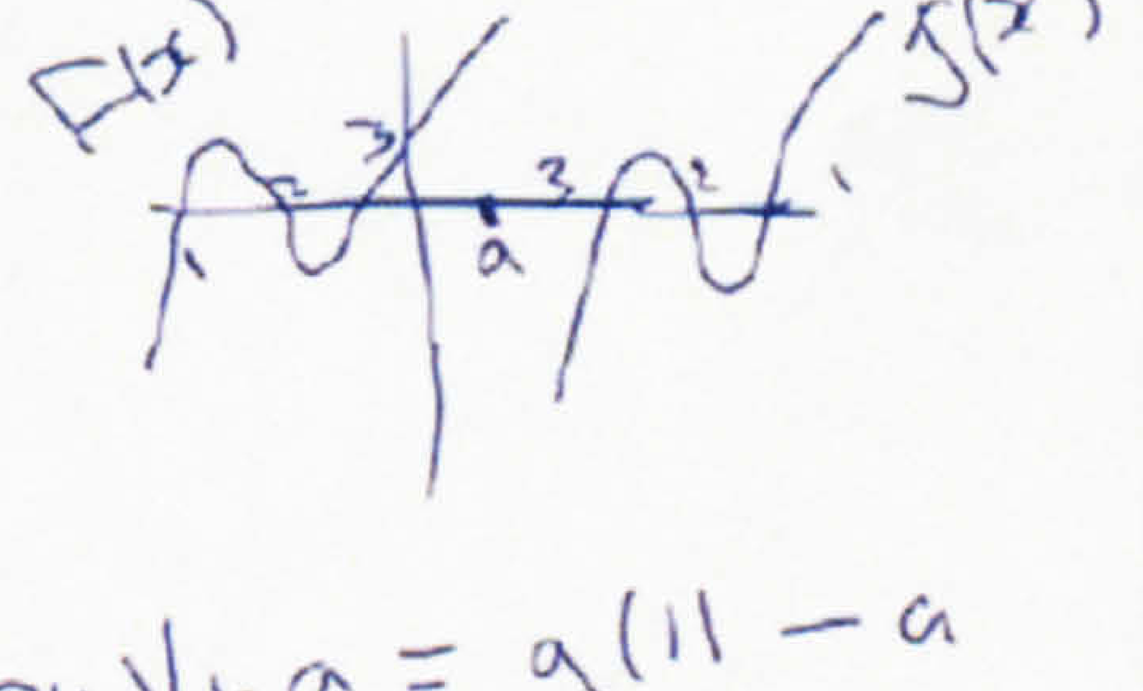
$r_1' = a + |a - r_1|$

$r_2' = a + |a - r_2|$

$r_3' = a + |a - r_3|$

WHERE r_1, r_2, r_3 ARE ROOTS OF $f(x)$.

P16: correct: graphic/algebraic



$|F(1)| + a = g(1) - a$

$g(1) = |F(1)| + 2a$

P17: not that correct: numeric/algebraic

For x^3: The solutions for G(x) are double the x co-ordinate for the point of rotation

If a constant is added (x^3+1) then an extra value appears added to the solution ($G(x) = 5$ for x^3+1)

Written inferences for the Root task in Interactive

P5: correct: graphic

The point (a, 0) defines a vertical symetry axe. For each root of f(x), g(x) has a root which is the symetric point relative to axis vertical defined by (a, 0).

P6: correct: algebraic

New roots after rotation can be ~~per~~ calculated as

a0 − (root − a0) = new root position

P11: correct: graphic

1.2

1.5 → 2.7 [illegible]

1.2

The roots are mirror about x = 1 but [illegible] [illegible]

The roots are mirror images about x = ~~[illegible]~~ — they — the graph is ~~a~~ rotated

P18: correct: numeric/algebraic

a = 0		a = 1	
f(x) →	g(x)	f(x) →	g(x)
2.6	−2.6	2.6	−0.6
−3	0	−3	2
0	3	0	5

when f(x) is rotated 180° about (a, 0) the roots of the new function are (−x + 2a, 0) where x is the value of the original root.

P13: correct: graphic

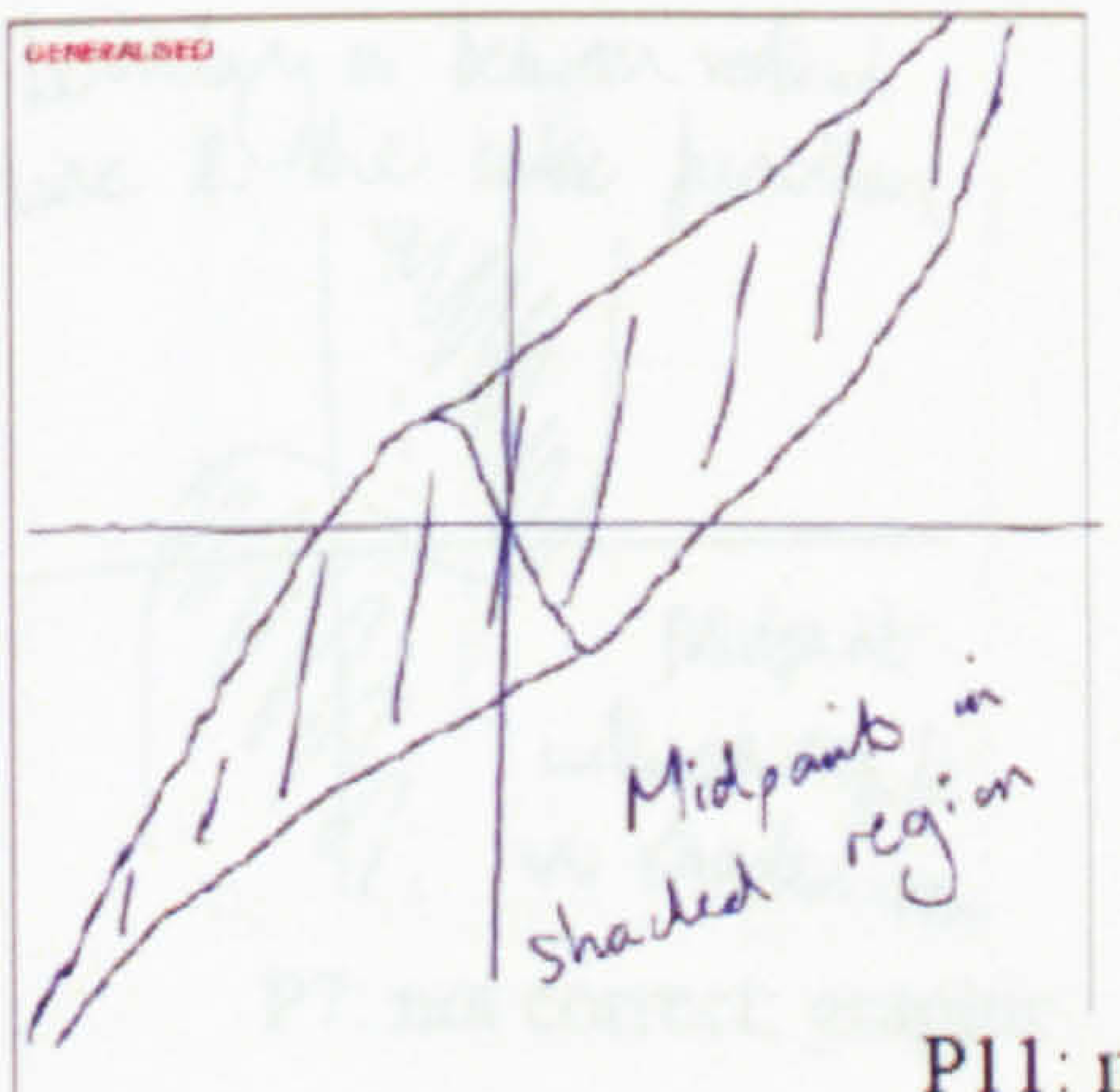
The point of rotation acts as a vertical mirror that reflects the ~~roots~~ onto the other side of the mirror.

P14: correct: graphic

The roots of the rotated function and the original function are symmetric about the centre of rotation.
This is because the centre of rotation is only allowed to be on the x-axis.

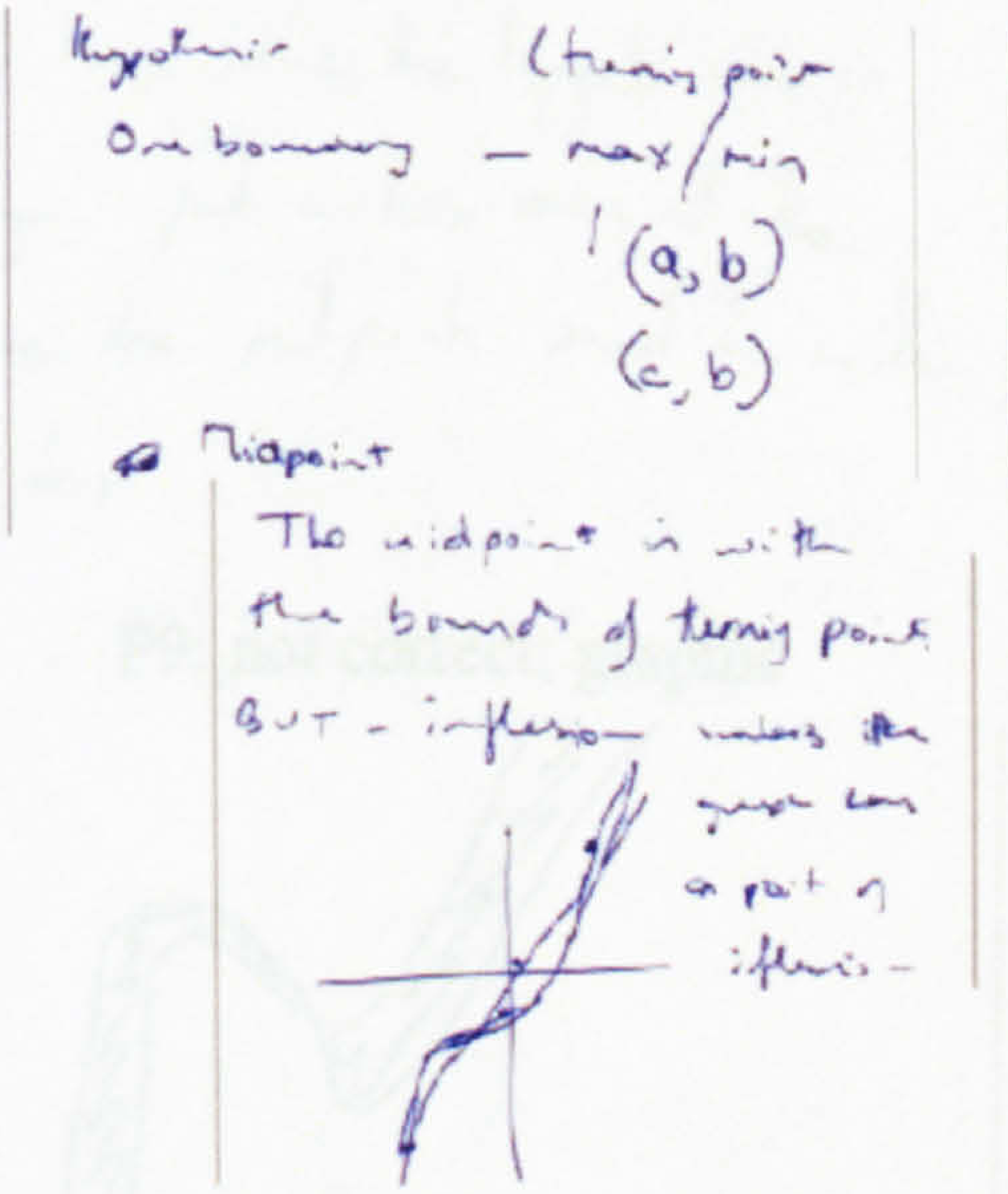
Written inferences for the Chord task in Static

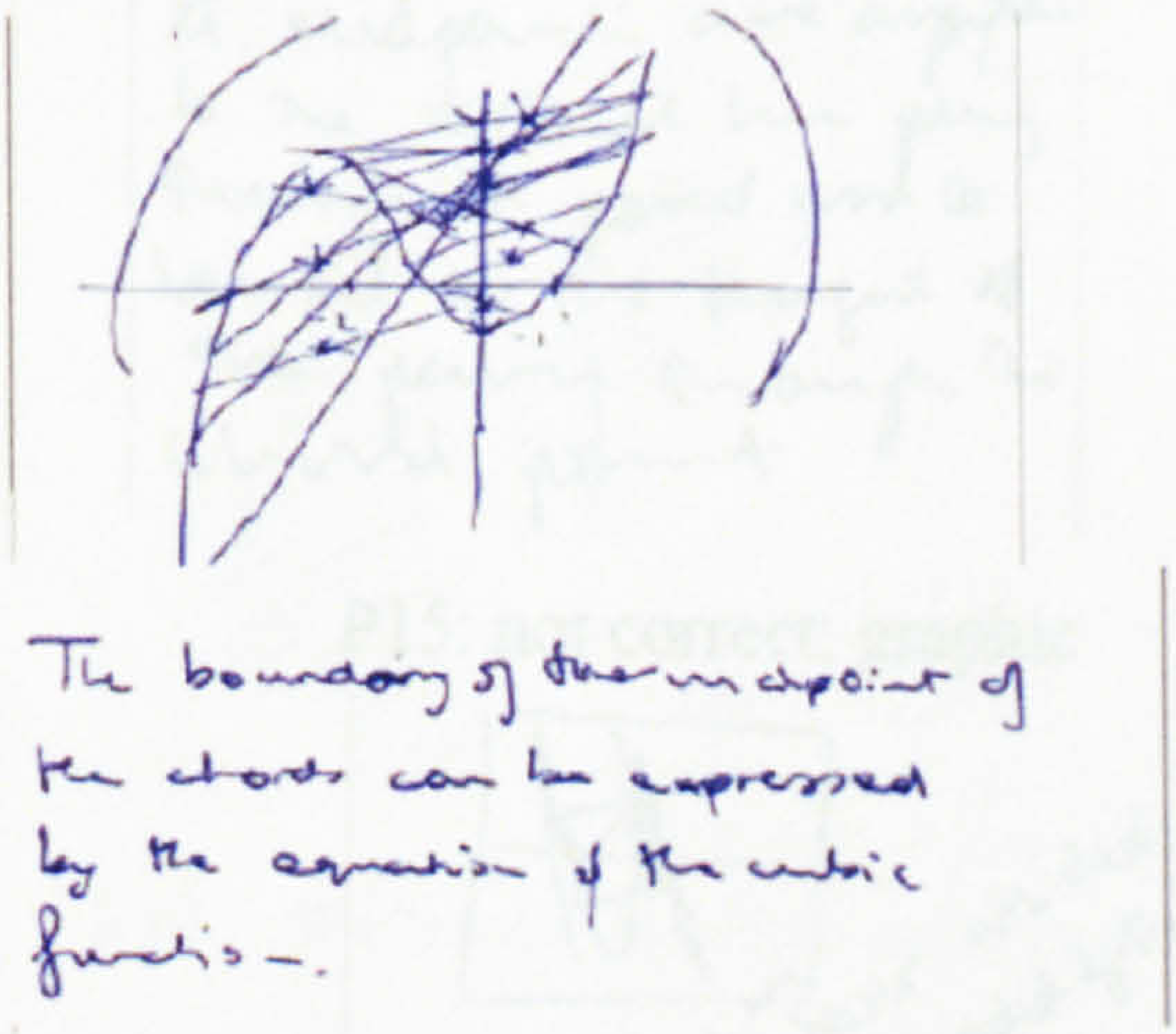
P1: not correct; graphic



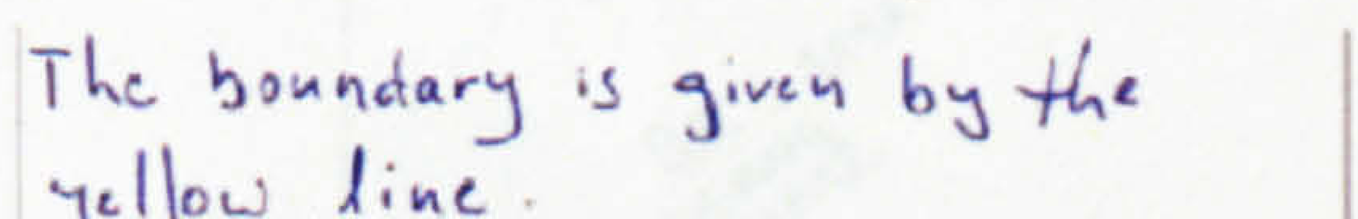
P2: graphic (did not finish on time)

P11: not correct; graphic

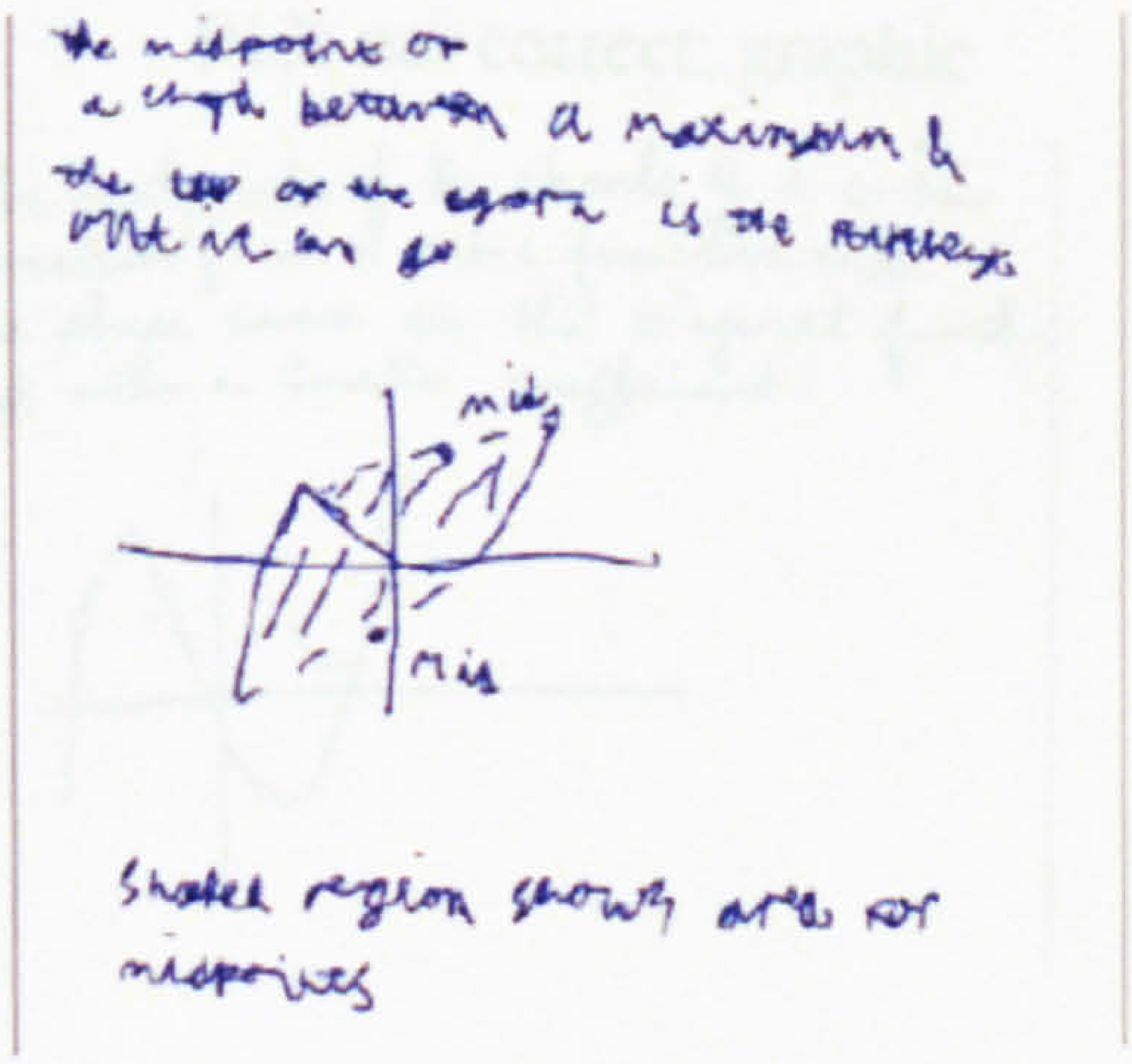




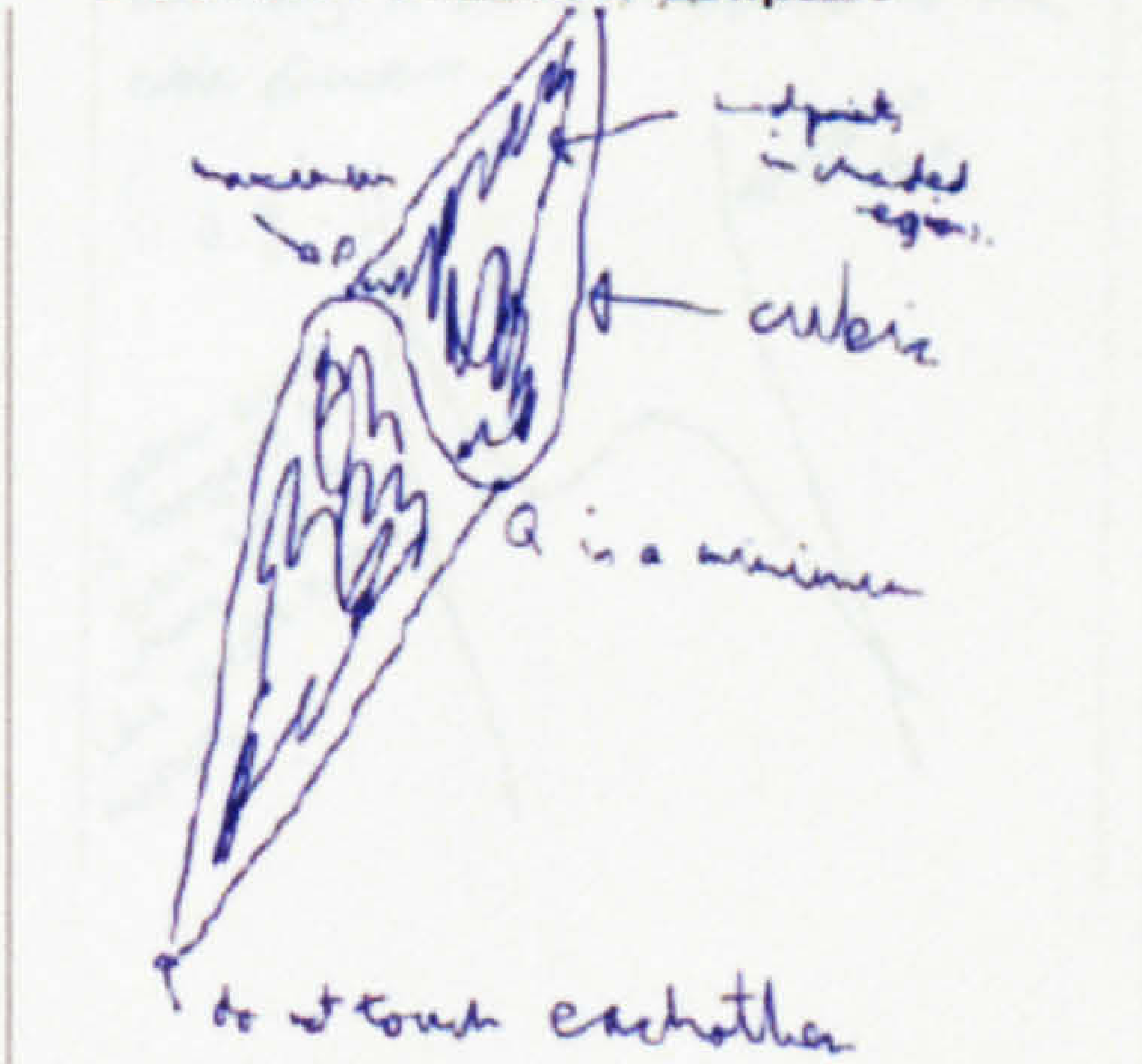
P14: not correct; graphic



P17: not correct; graphic



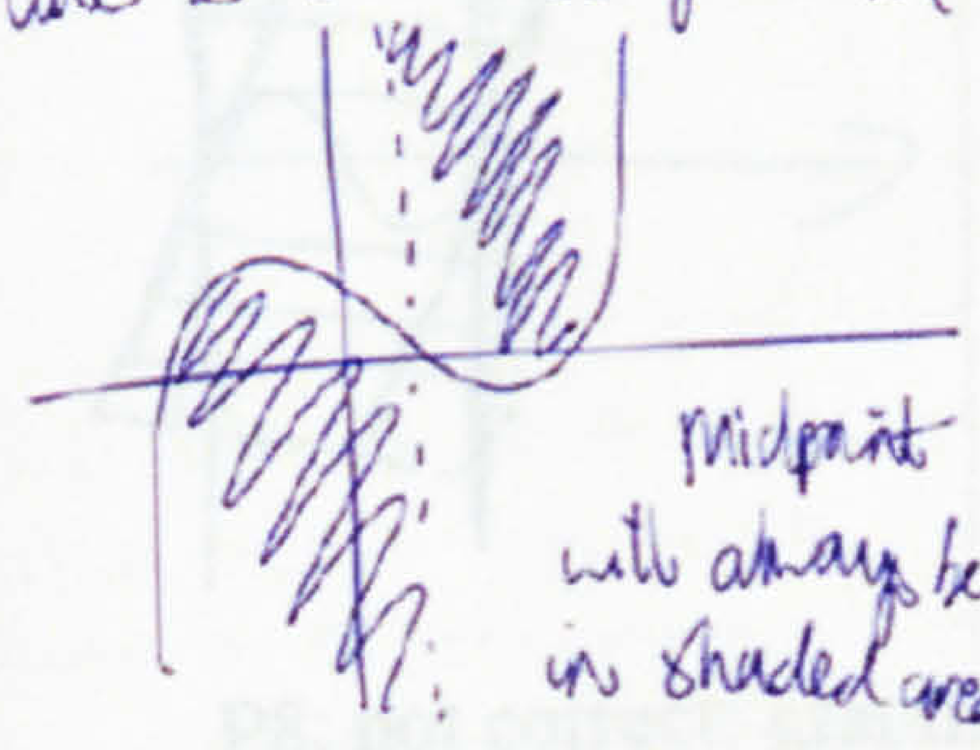
P18: not correct; graphic



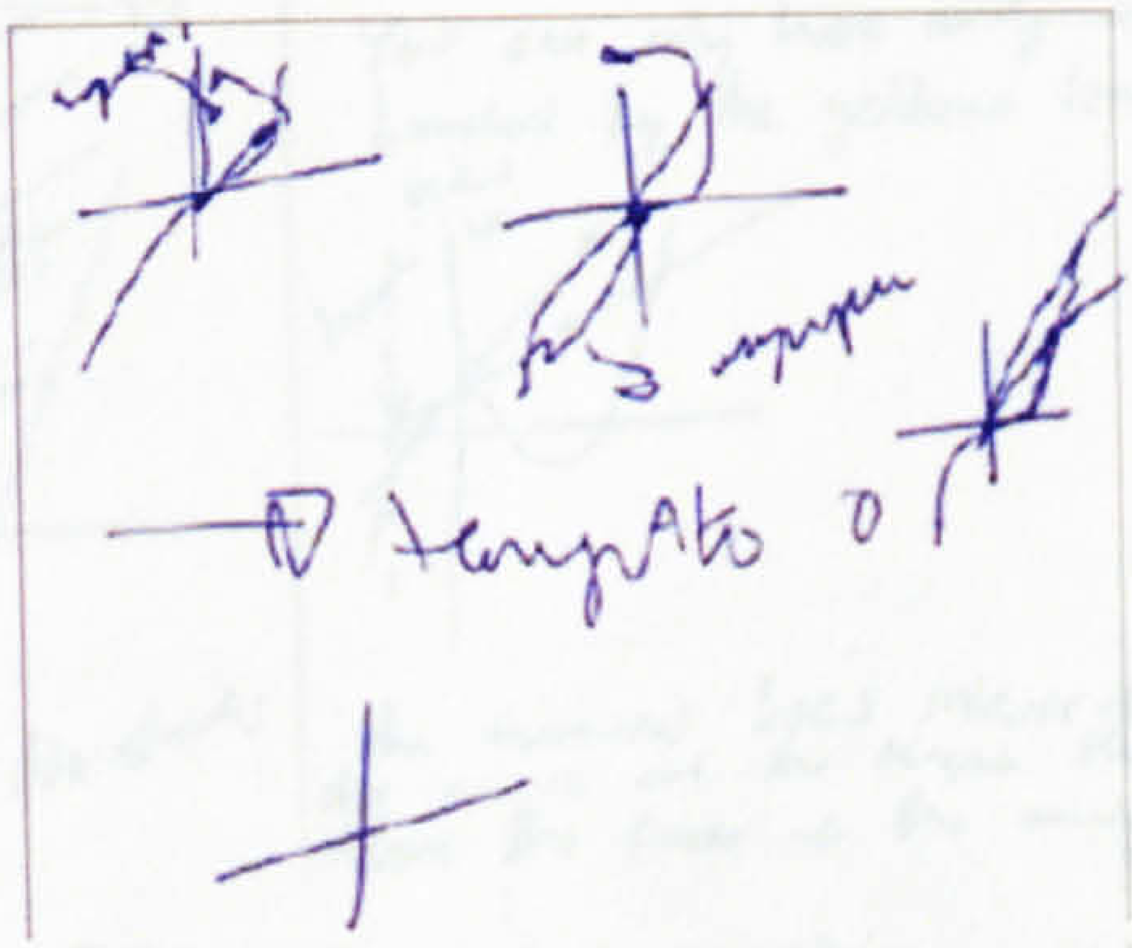
Written inferences for the Chord task in Dynamic

P3: not correct: graphic

boundary is between vertical line & the cubic function



P4: almost correct: graphic



P7: not correct: graphic

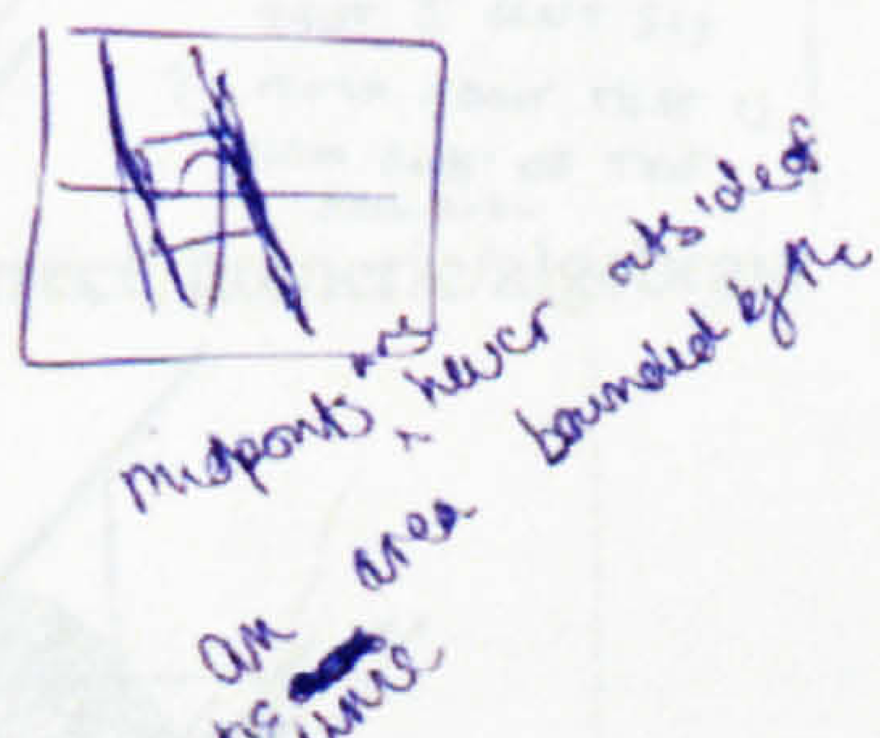
Don't know where the tangents come in. If you put a box around the curve the midpoints must lie within that box

P10: not correct: graphic

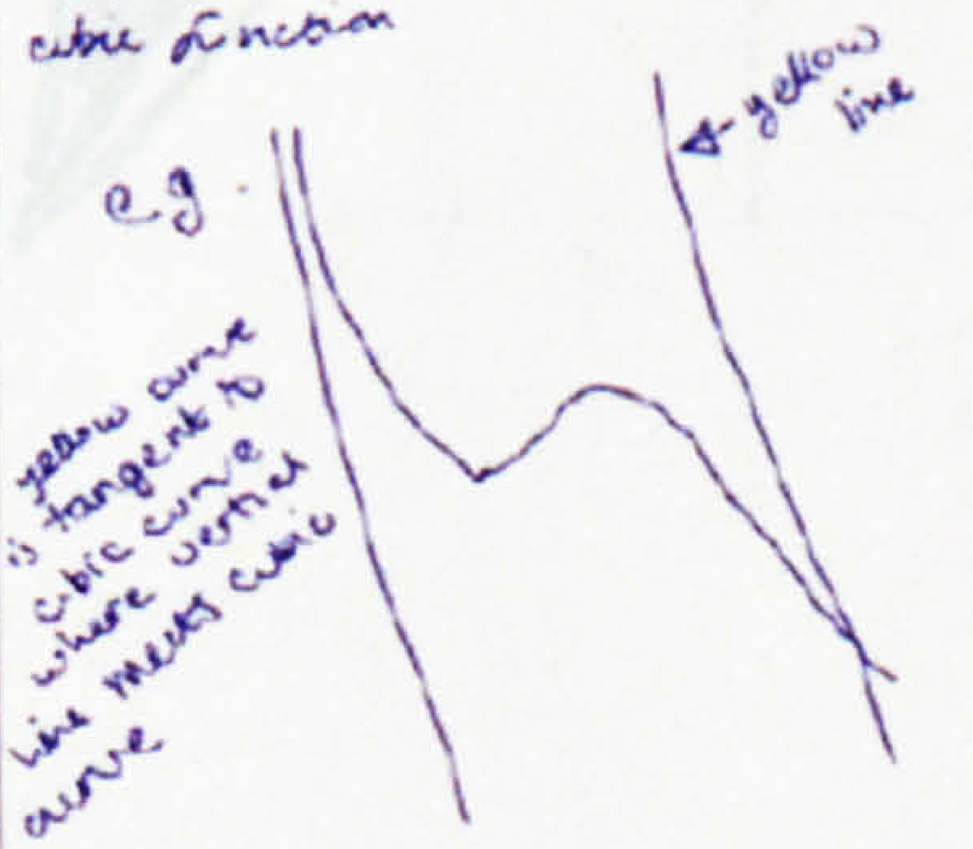
The boundary of the set of midpoints are asymptotic to the vertical line passing through the point and is bounded by the tangent of the line passing through the chord point

P9: not correct: graphic

P15: not correct: graphic

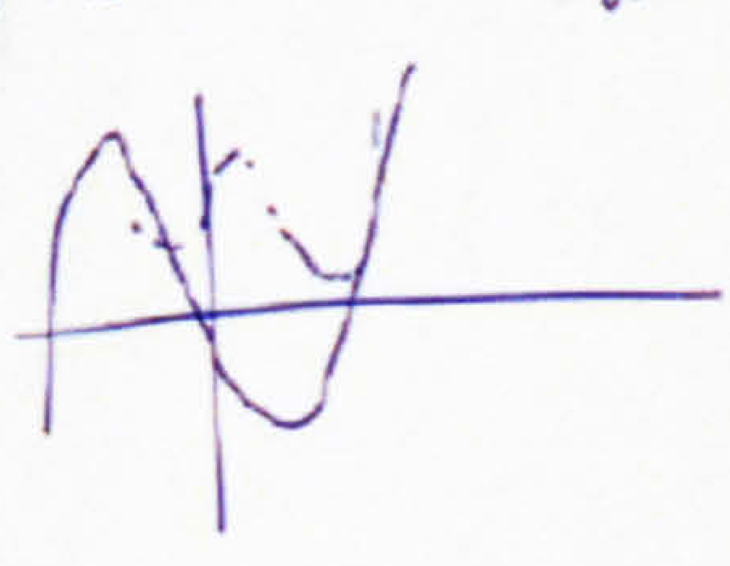


boundary of set of midpoints is the cubic function



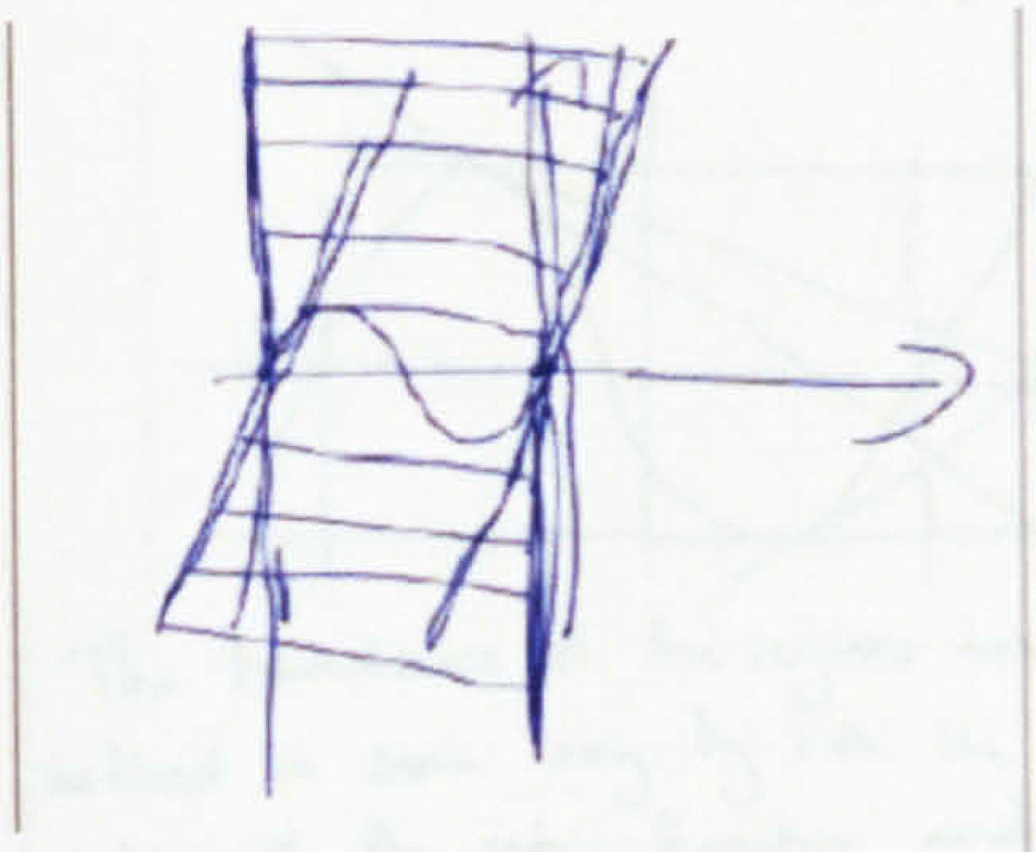
P13: not correct: graphic

The midpoints of the chords of a cubic function form a cubic function with the shape same as the original function but with a smaller coefficient.

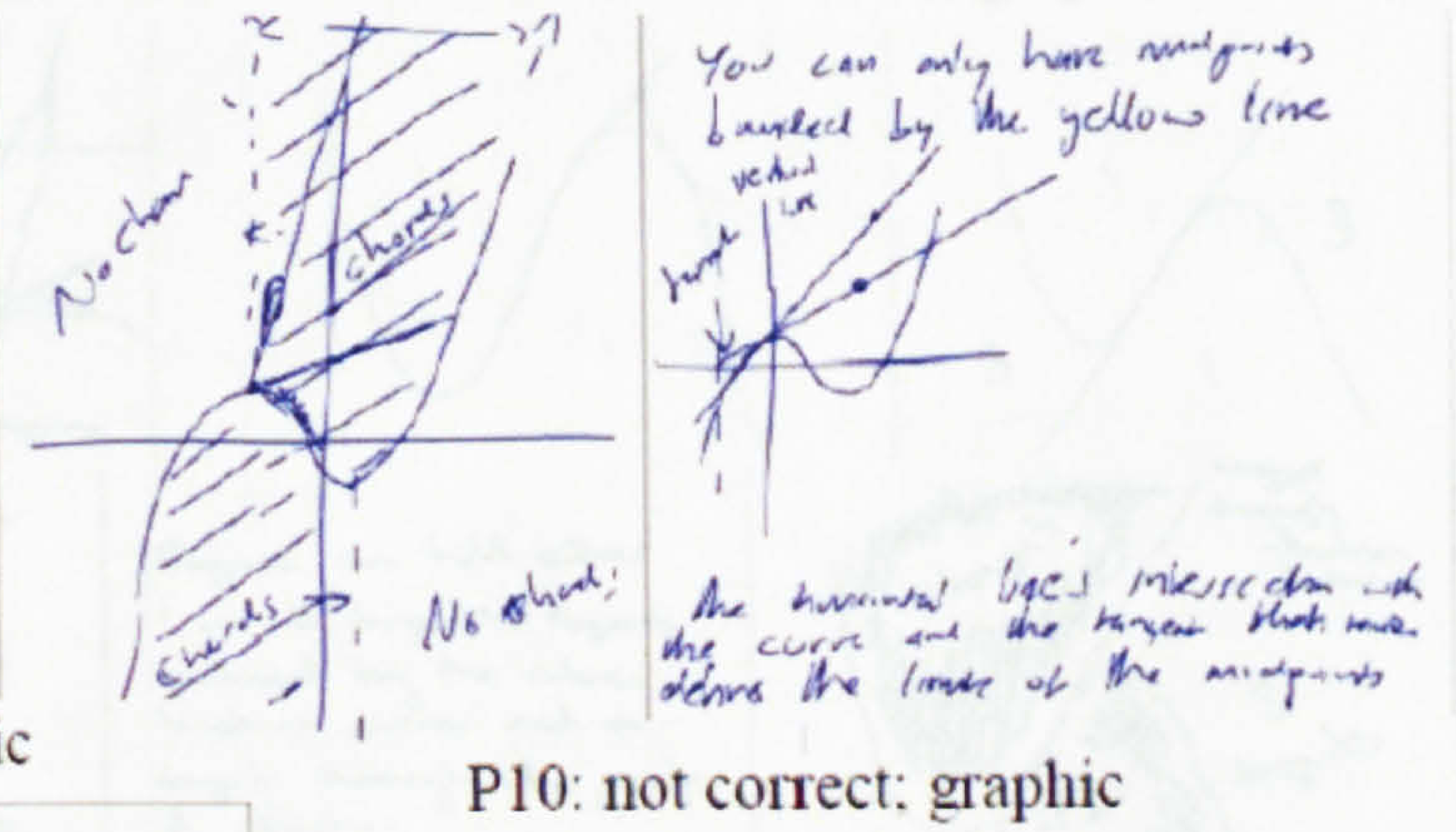


Written inferences for Chord task in Interactive4

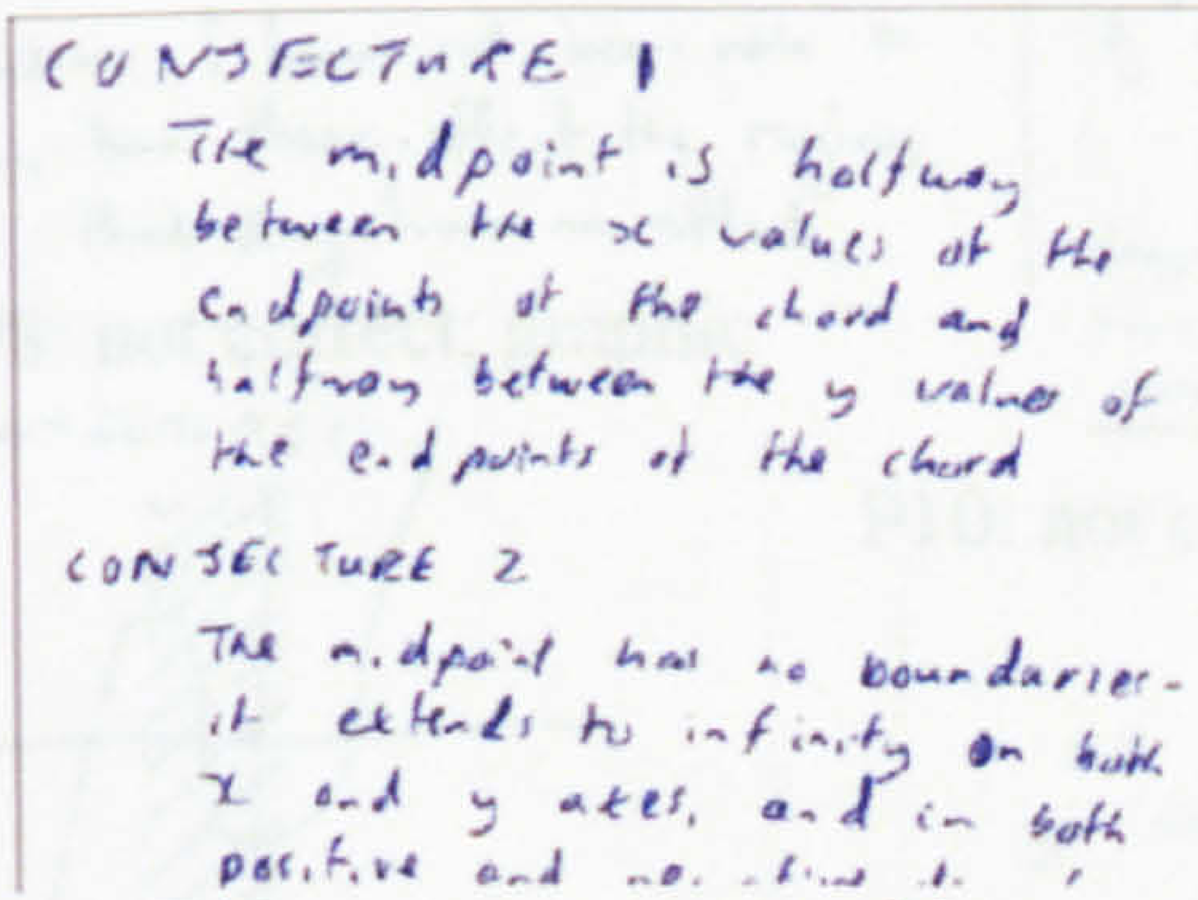
P5: not correct: graphic



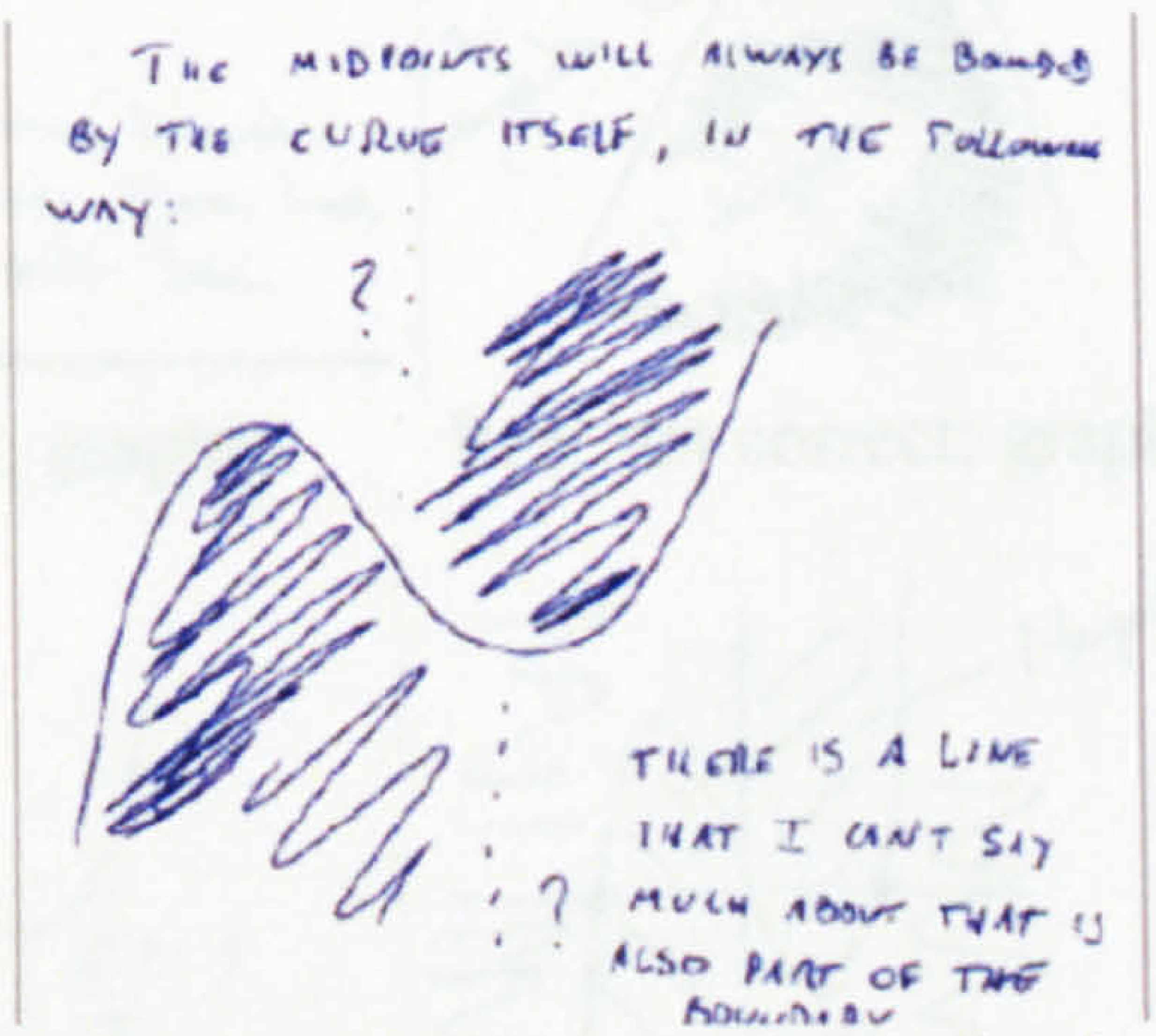
P6: not correct: graphic



P8: not correct: graphic

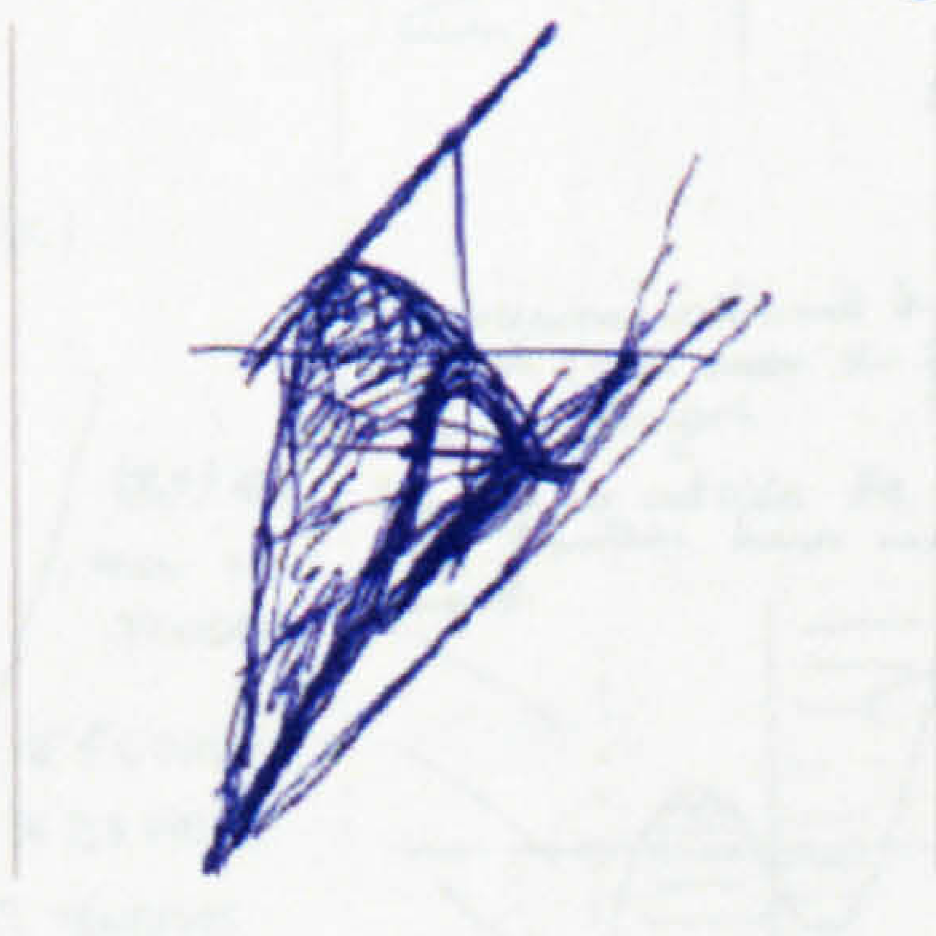


P10: not correct: graphic



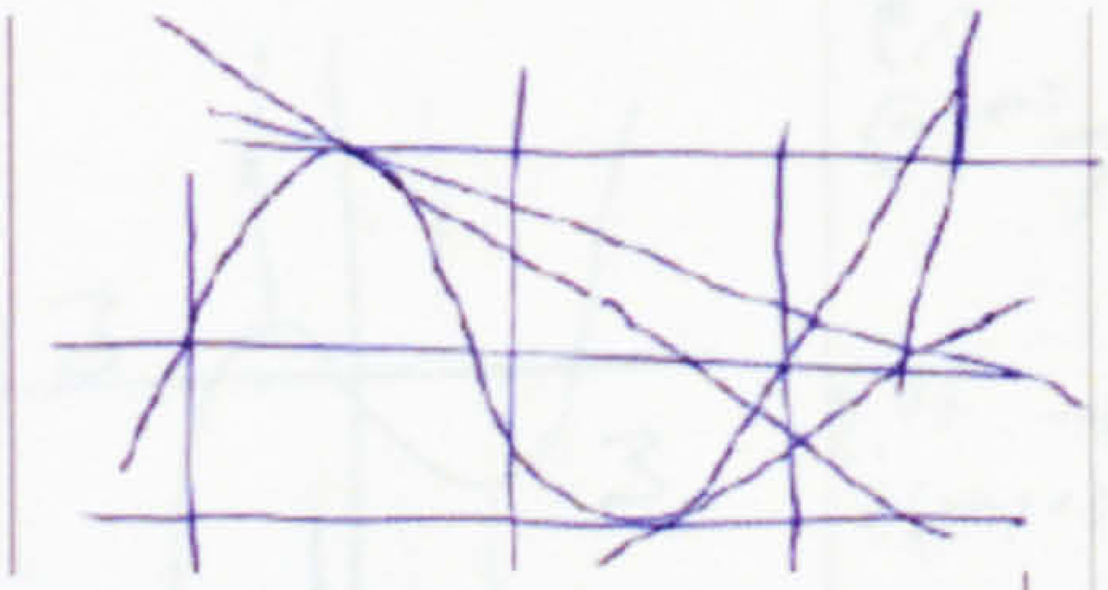
P12: not correct: graphic

P16: not correct: numeric/algebraic



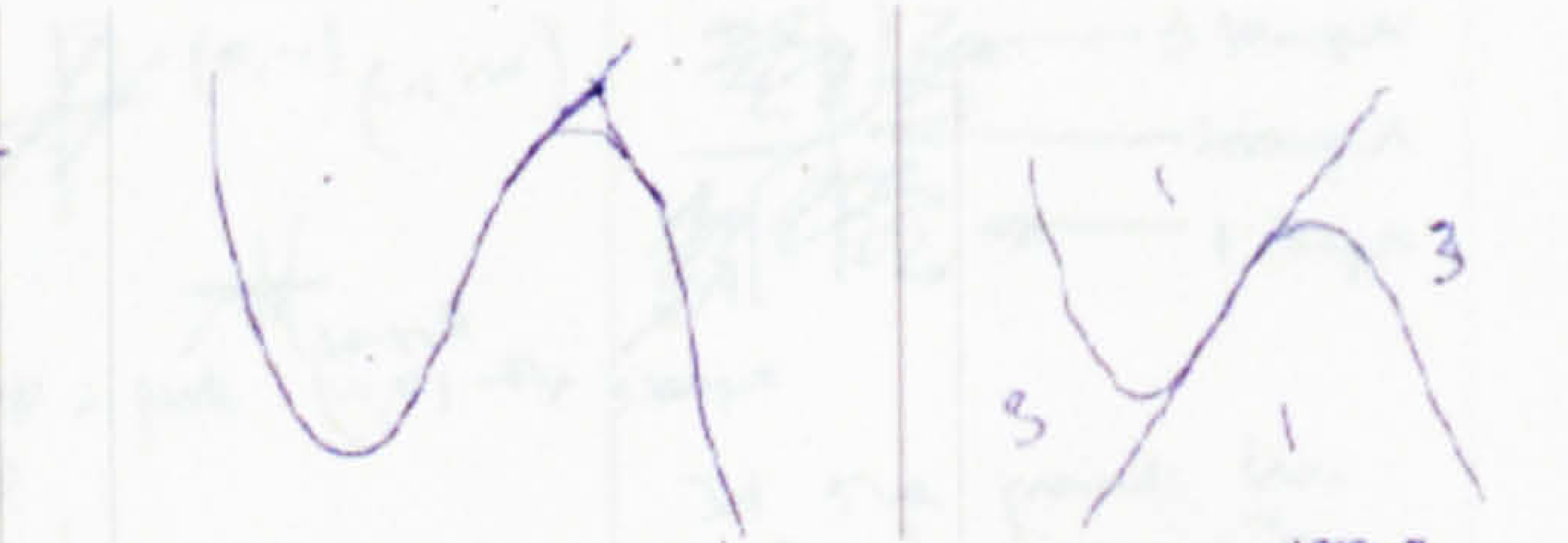
Written inferences for the Tangent task in Static

P1: not correct: graphic



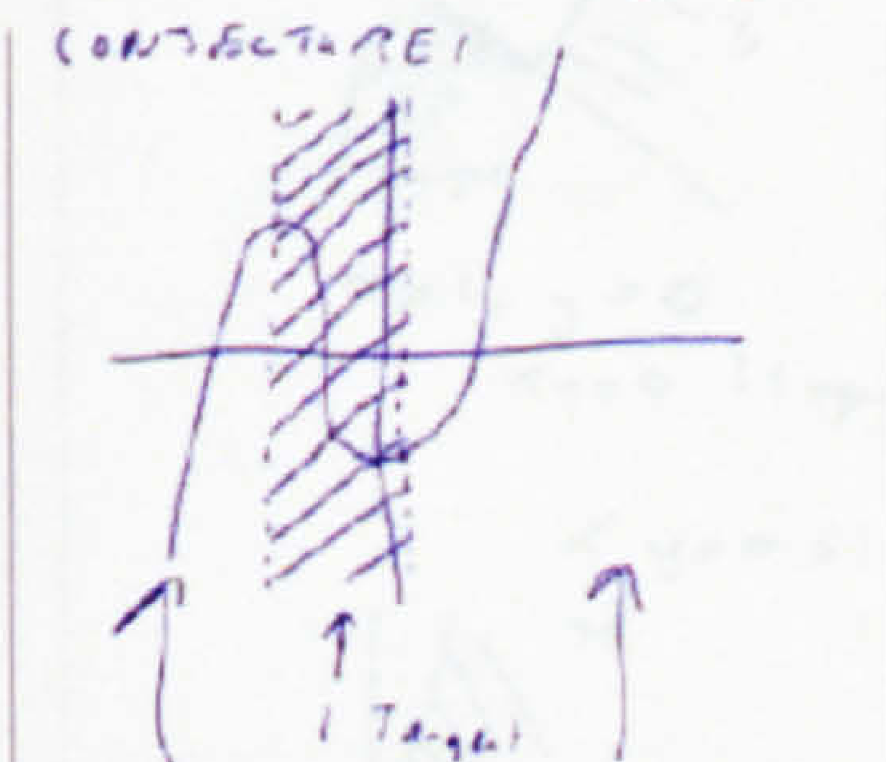
The boundaries of the regions are defined in some way by both the vertices of the cubic function and the outermost lines of the cubic function. I have not been able to define how these effect the regions, just that they have an effect.

P2: correct: graphic

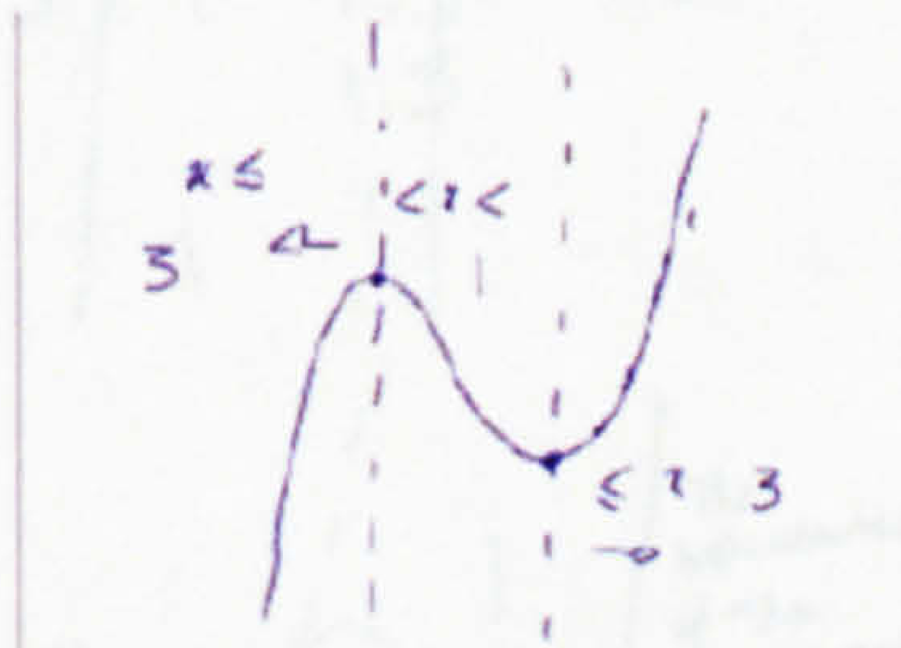


Regions can hold either 1 or 3 tangents. Regions enclosed by the cubic function curve and a tangent through its point of inflection have one tangent. Regions outside these has have 3 tangents. See over leaf

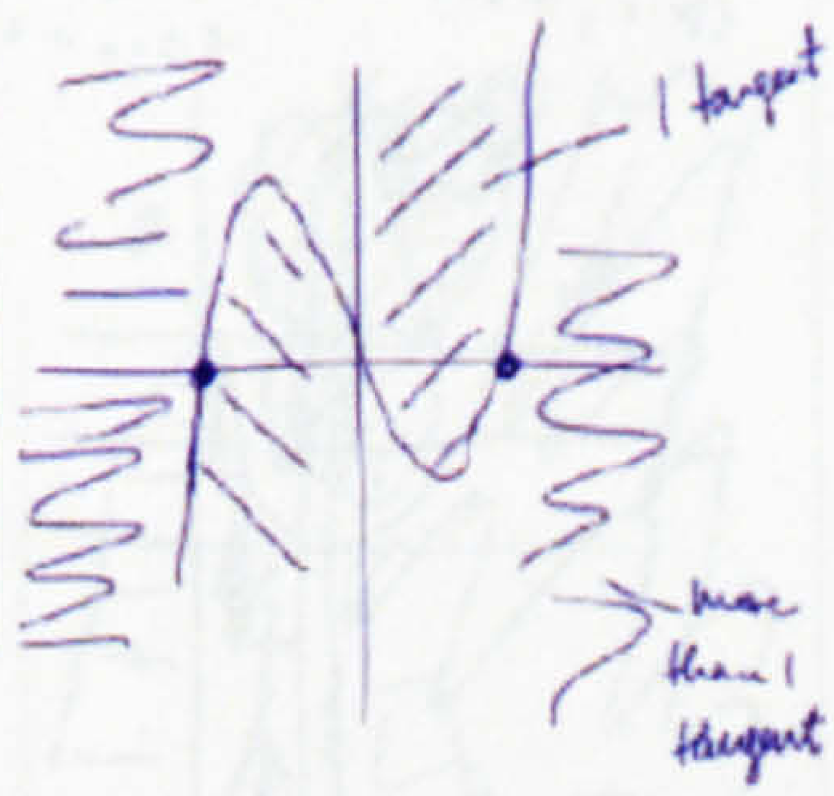
P8: not correct: graphic



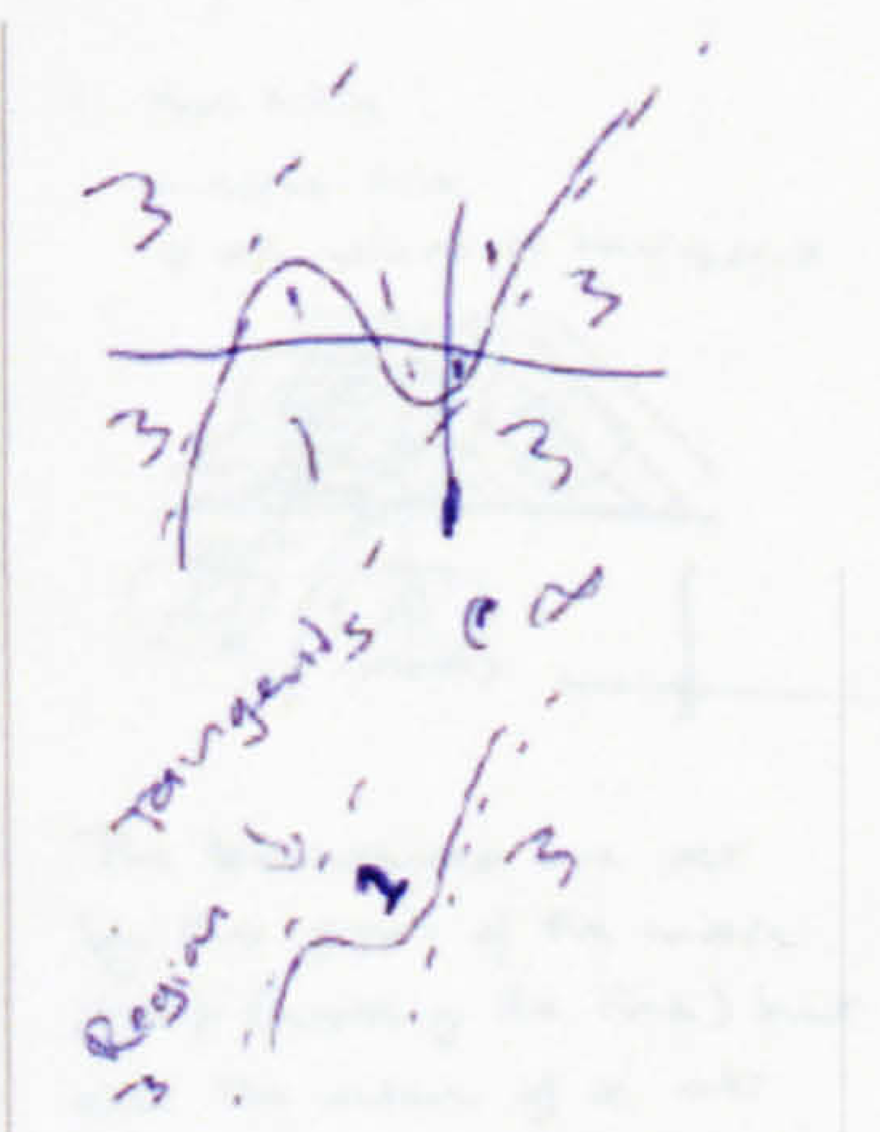
P10: not correct: graphic

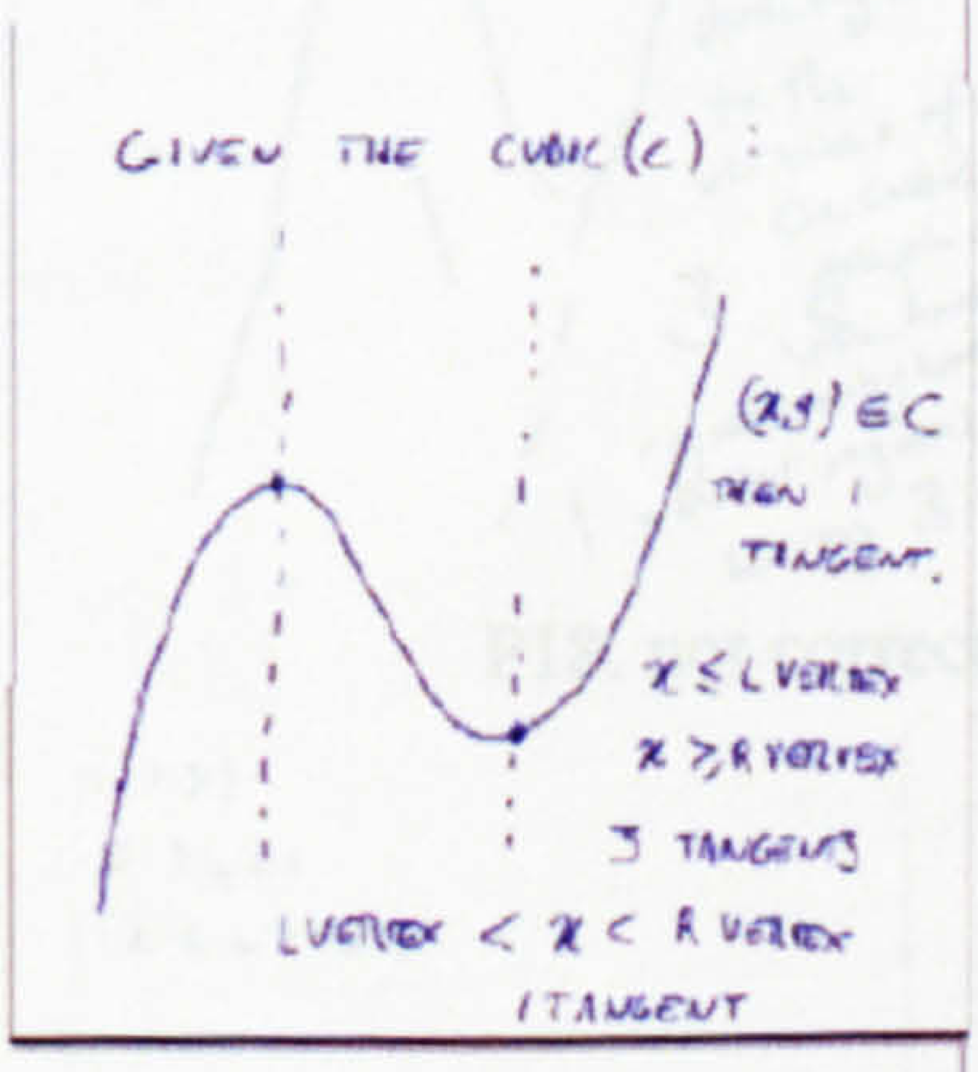


P13: not correct: graphic

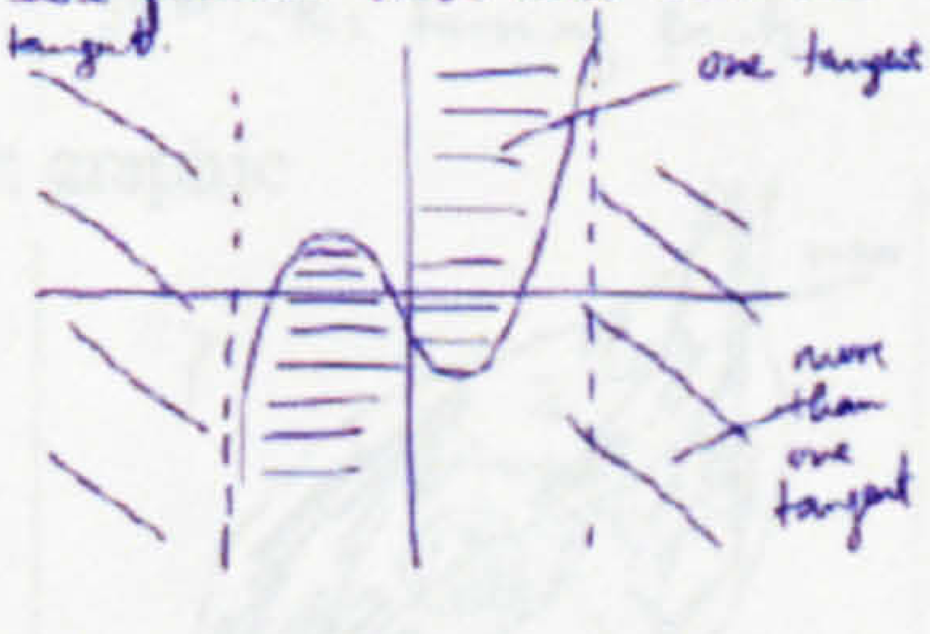


P16: not correct: graphic

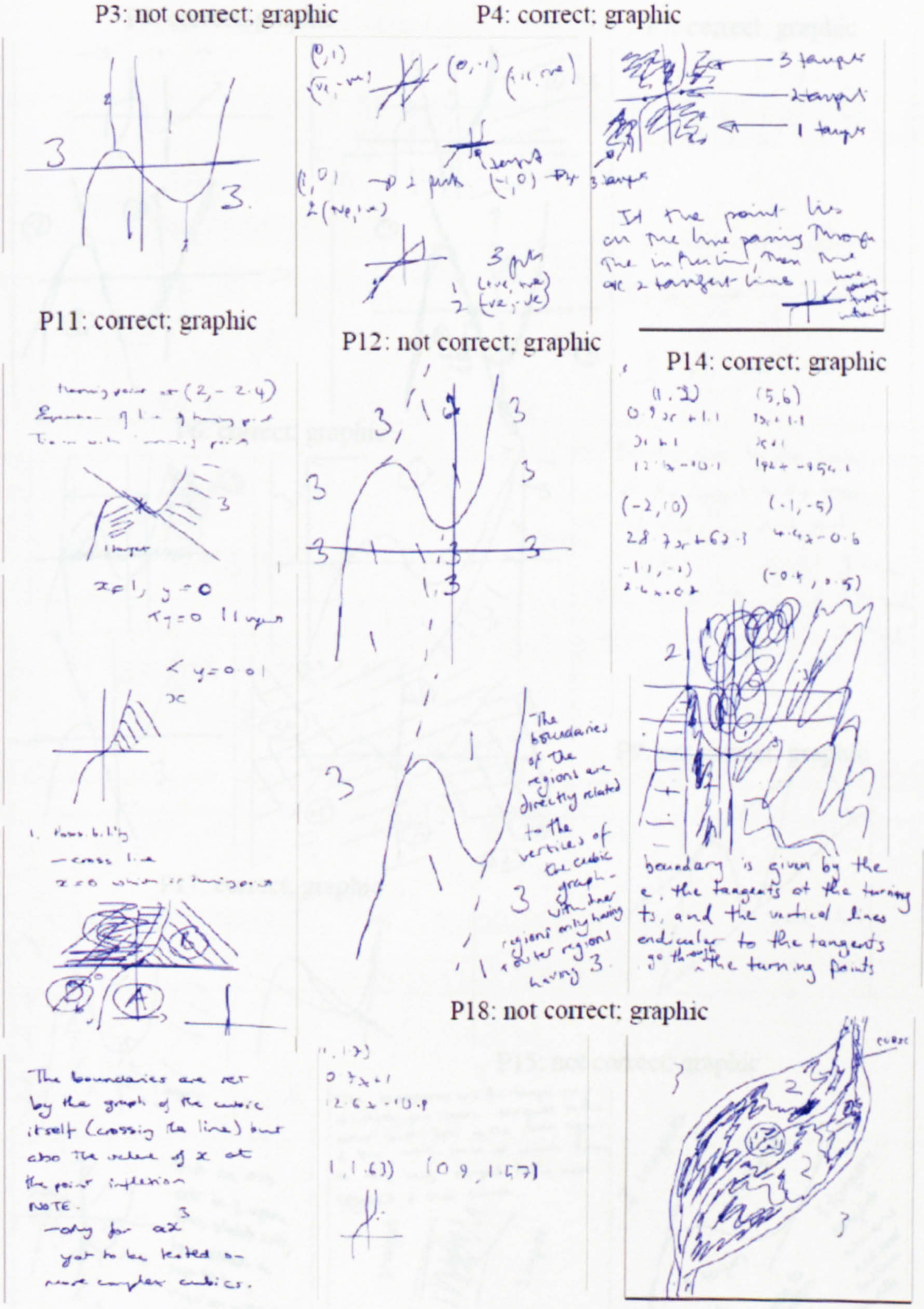




The regions contained by the cubic function (area under the graph) have only one tangent. The regions outside the curve of the cubic function have more than one tangent.

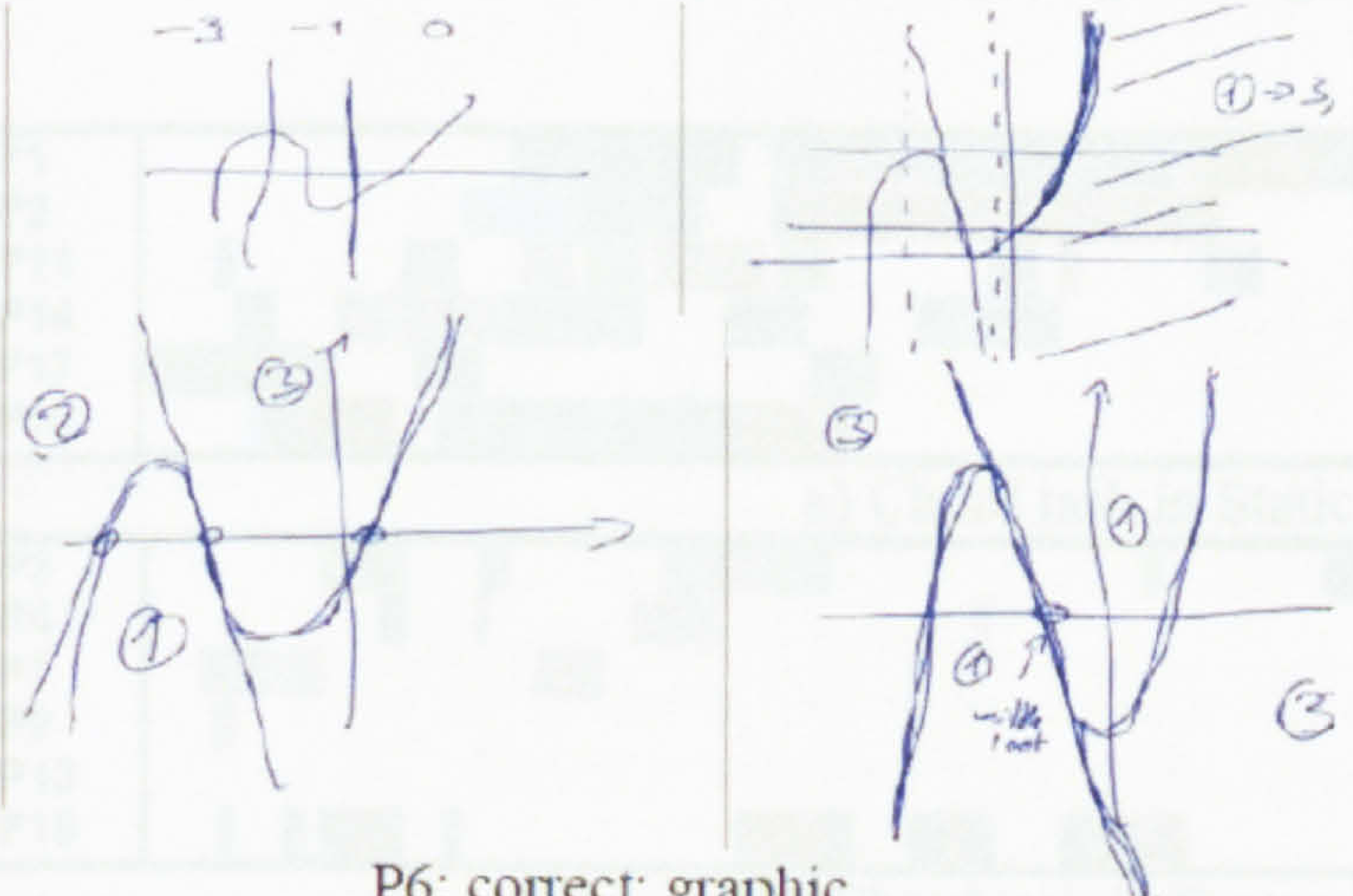


Written inferences for the Tangent task in Dynamic

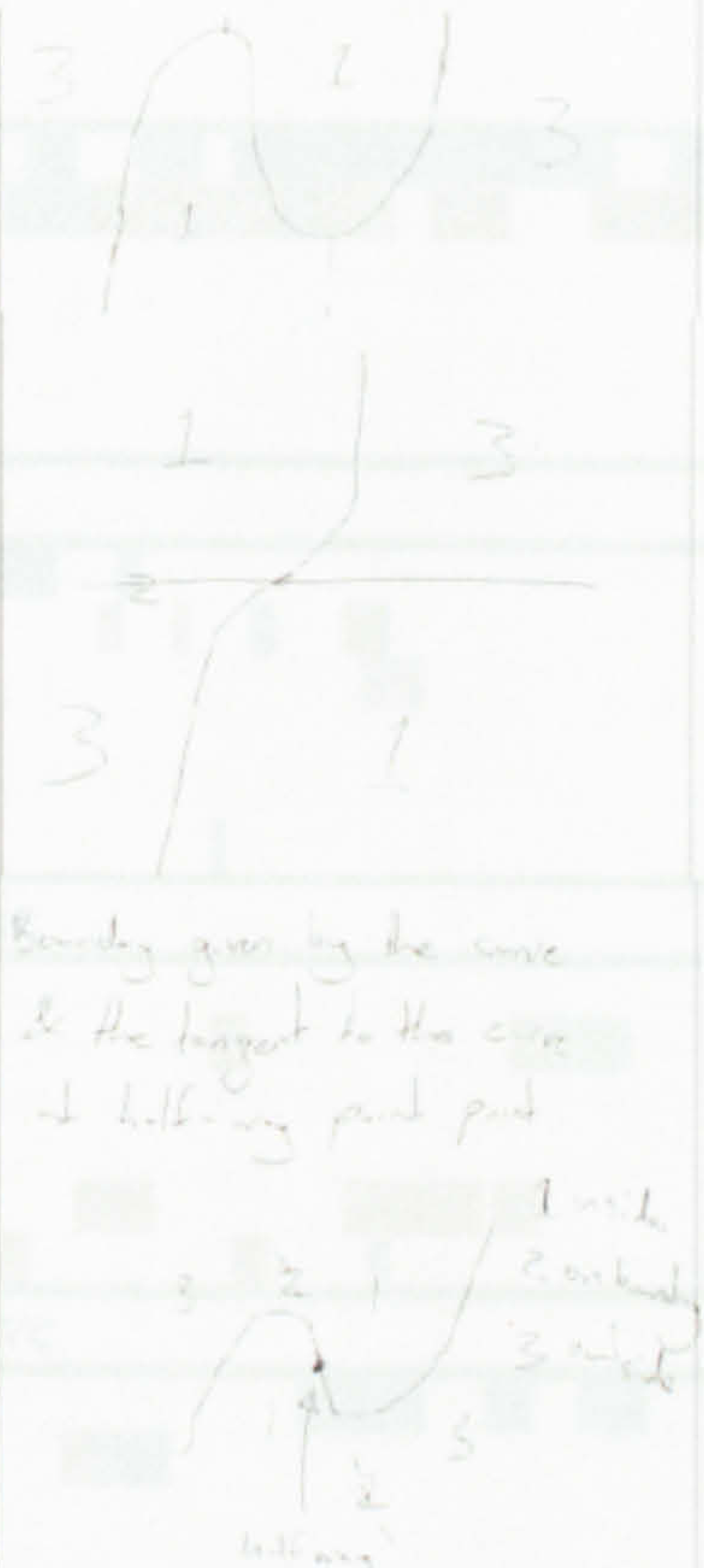


Written inferences for the Tangent task in Interactive

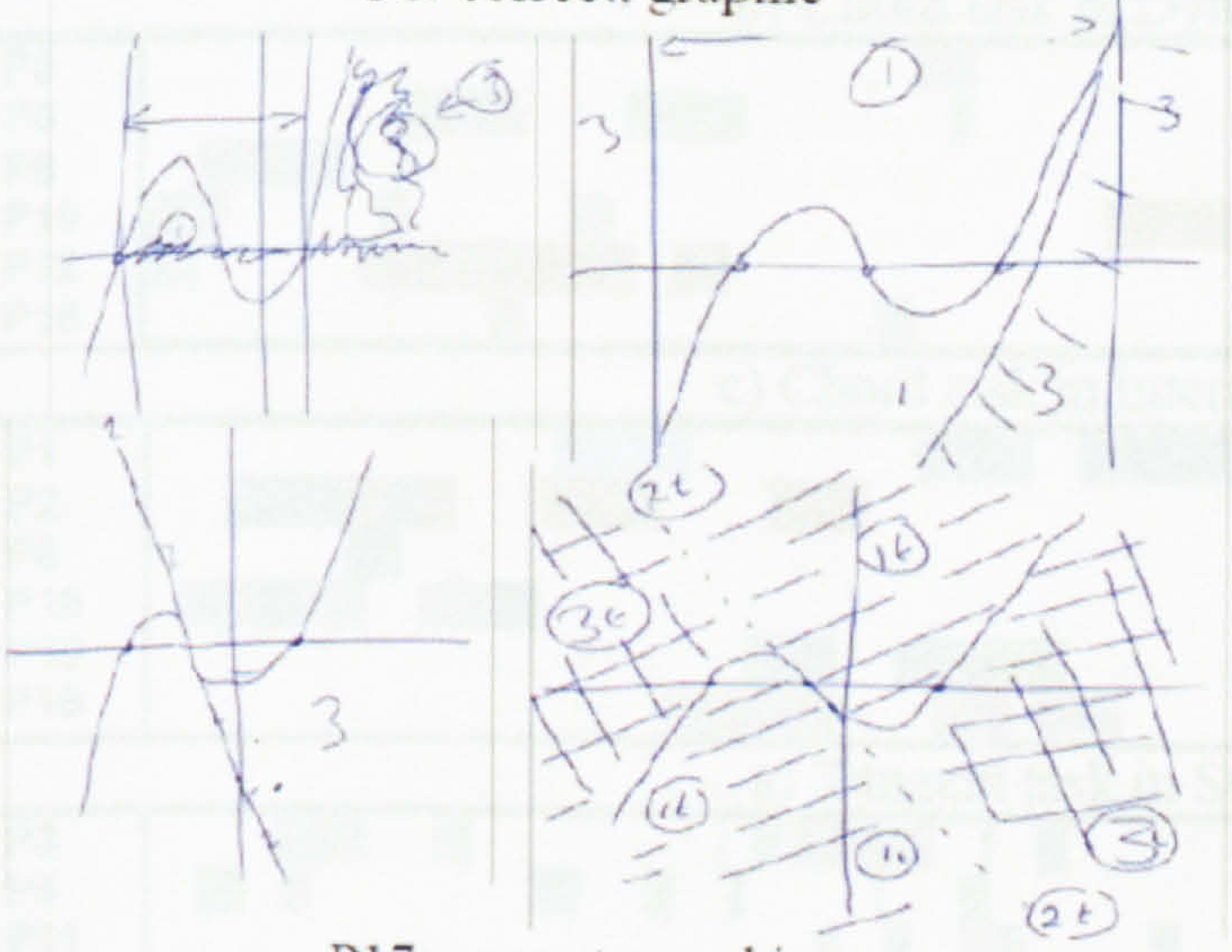
P5: correct: graphic



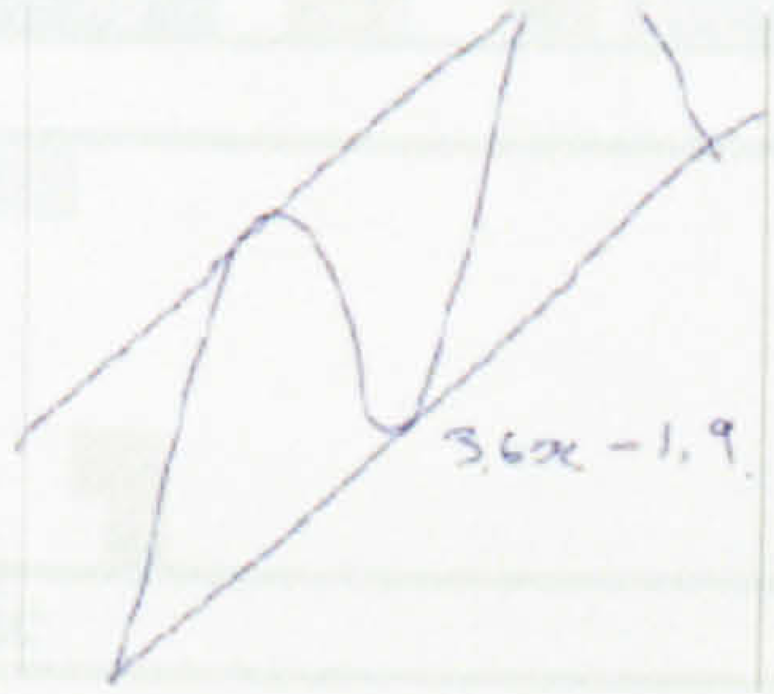
P7: correct: graphic



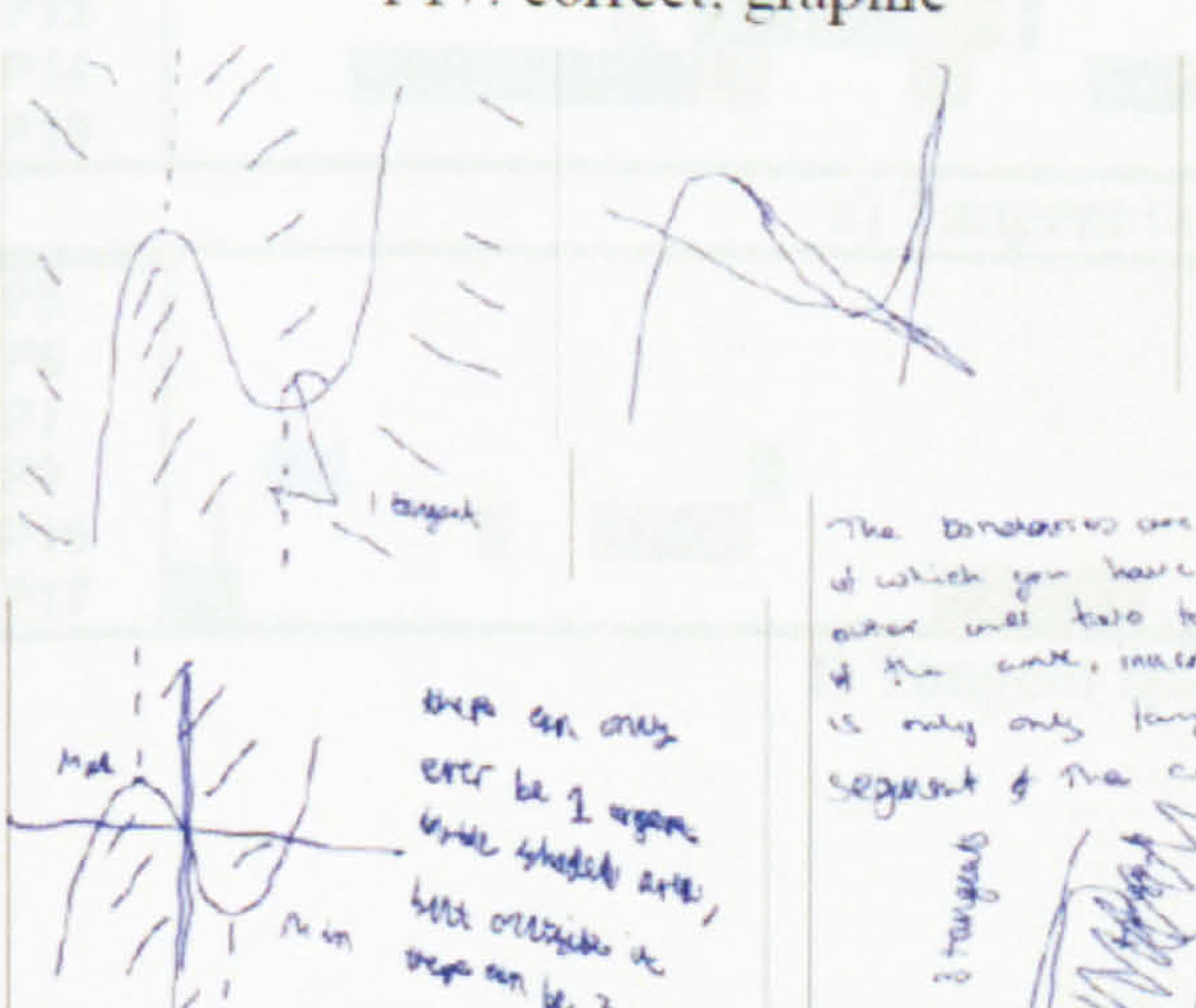
P6: correct: graphic



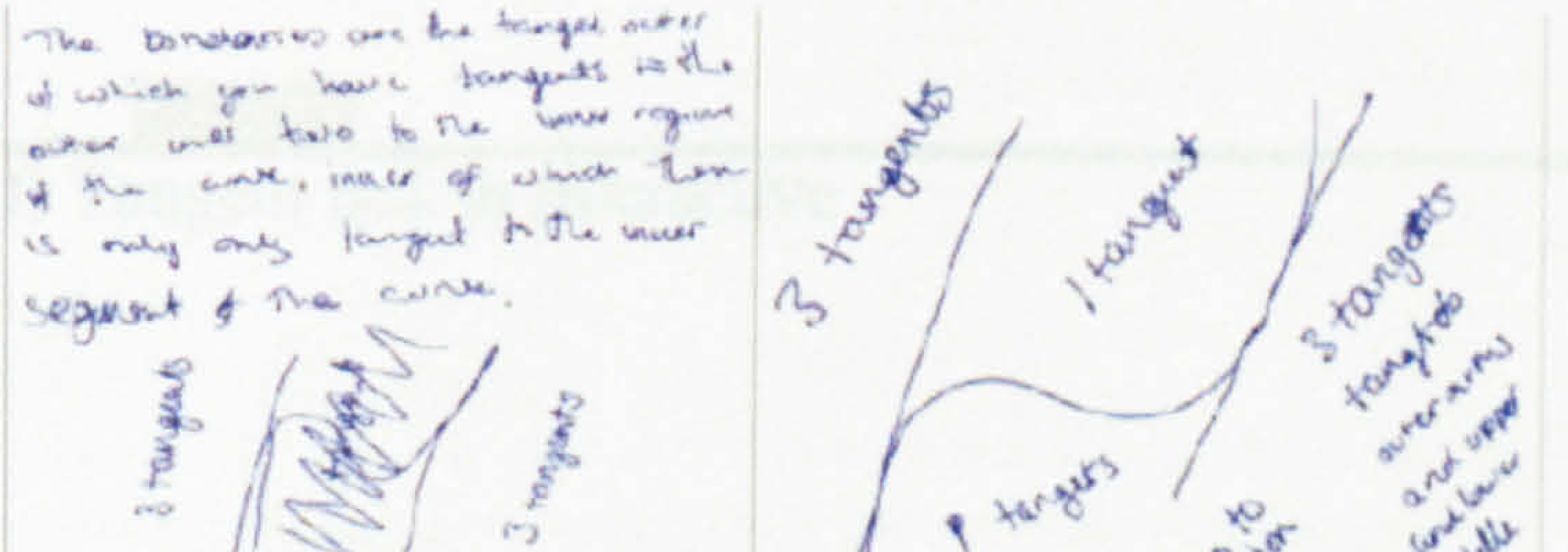
P9: not correct: graphic



P17: correct: graphic

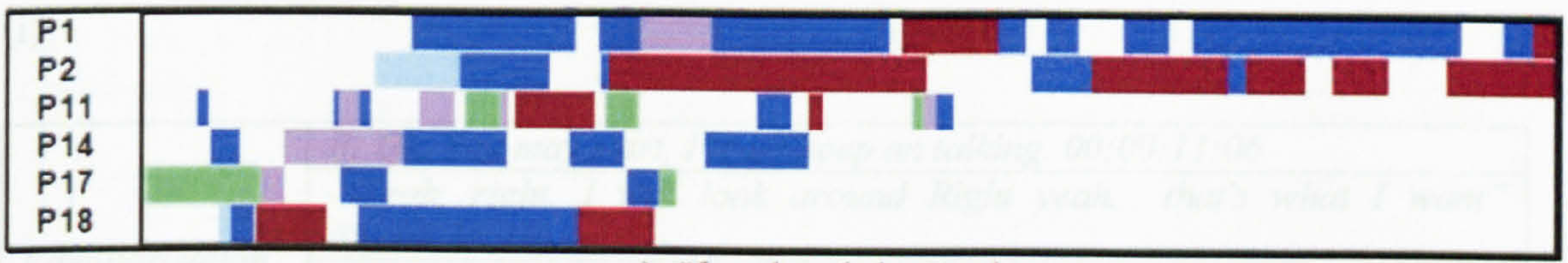


P15: not correct; graphic

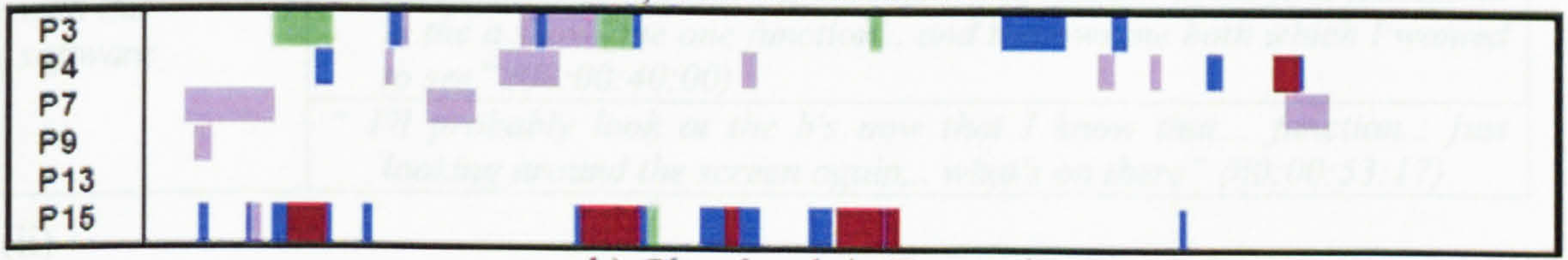


APPENDIX F IMAGINING STRATEGIES FOR THE CHORD AND TANGENT TASKS

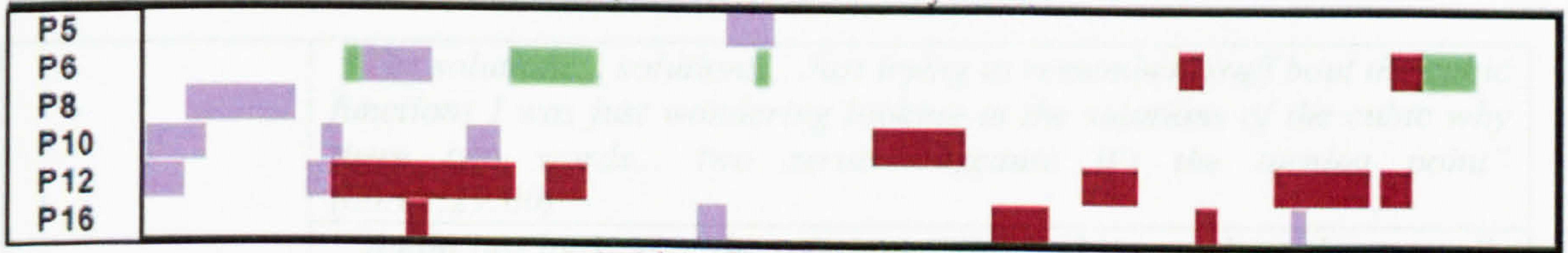
Timelines analysis of imagining strategies for two tasks



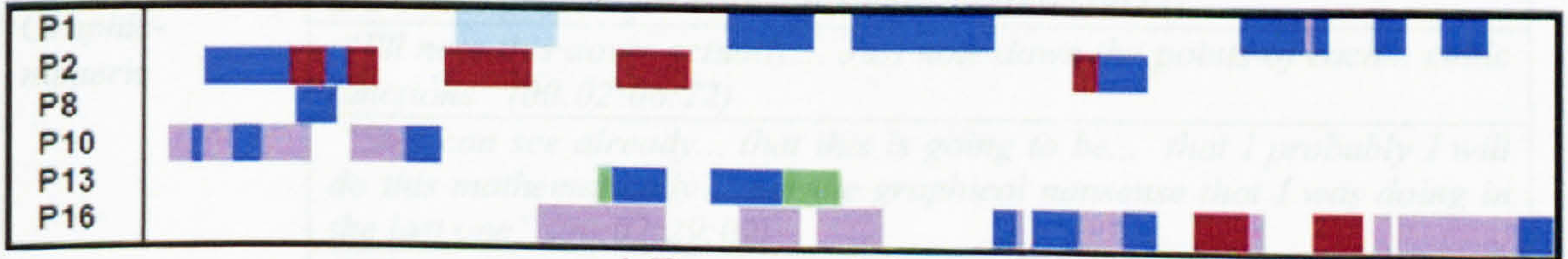
a) Chord task in Static



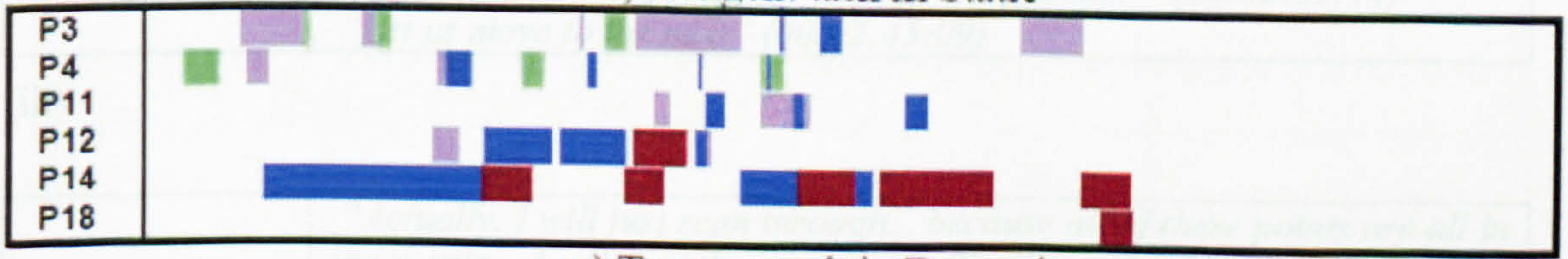
b) Chord task in Dynamic



c) Chord task in Interactive



a) Tangent task in Static



a) Tangent task in Dynamic

f) Tangent task in Interactive

APPENDIX G CHANGE IN STRATEGIES

P2's chosen strategies in the Root task in Static

(i)

Familiarisation with the software	*R: Ok. You may start. Please keep on talking. 00:00:11:06*
	"Yeah, right. I will look around Right yeah... that's what I want" (00:00:16:19)
	"I'm just looking to see... right... that it shows me... in the first function... in the a shows me one function... and b shows me both which I wanted to see" (00:00:40:00)
	" I'll probably look at the b's now that I know that... function... just looking around the screen again... what's on there" (00:00:53:17)

(ii)

Graphic-numeric	*"The solutions... solutions... Just trying to remember stuff bout the cubic functions I was just wondering looking at the solutions of the cubic why there two words... two zeros... because it's the turning point" (00:01:29:00)*
	" Again just looking at the solutions... just looking at where they actually are on the graph...for both functions" (00:01:39:14)
	"I'll note this down actually... Just note down the points of each... cubic functions" (00:02:06:12)
	"So I can see already... that this is going to be... that I probably I will do this mathematically... not the graphical nonsense that I was doing in the last one" (00:02:29:02)
	"The coordinates of the point is rotated around..." (00:02:33:14)
	"Let us move to the next" (00:02:45:09)

(iii)

	"Actually, I will just scan through... because all of these points are all in the x axis... I will just have a quick flip through and see if that is the case for each one of these" (00:03:02:14)
	"It looks like it is" (00:03:05:10)
	"Now, look at the question again on the pad... Oh yes it says a not" (00:03:24:18)
	" So the same y value yeah... it crosses the x axis so the same y value yeah" (00:03:29:17)
	"It shows that I did not read the question well (laughs)... Hang on... I'm just reading the question" (00:03:43:12)
	"I'm gonna look on the first one" (00:03:57:08)
	"What I'm going to do now is just look at... what I've written down which is the solutions of these two cubic functions and the coordinates of the point which is rotated around" (00:04:11:21)

(iv)

Graph-wise	*"I'm just gonna have a look mathematically at err... what relationship is between those too and the... what I can already see Probably what it is gonna be... I'm just doing it in my head" (00:04:35:03)*
	"So I will just probably look at the first one and check my solution against the others... Ok I will stop looking at the screen" (Laughed) (00:04:45:22)
	"I'm just looking at what is on the screen again this first example... Err" (00:04:58:18)
	"Just looking at where those points are on the line" (00:05:03:01)
	" Making sense of what I'm gonna do mathematically" (00:05:09:06)
	"I don't like to see things... (Laughed)... So alright..." (00:05:21:08)
	R: Any inference so far? (00:05:31:14)

(v)

	"Yeah... not the actual cubic... but the actual points are mirrored... I think... around the rotation points" (00:05:41:24)
	" I'm just gonna look through each of the points of the first function" (00:06:08:16)

(vi)

Graphic-numeric	*"I will just work out the symmetry" (00:06:13:09)*
	"I'm trying to do it mathematically..." (00:06:23:15)
	"So... I'm just going to use one point..." (00:06:34:19)
	"The outermost point ... which is the last solution of f(x) and the first solution of g(x)" (00:06:44:01)
	" Ok... compare them" (00:06:51:12)
	(wrote) P1 (0, 0, 2.5) P2 (-5.5, -3, -3) a(-1.5, 0)x1 = 2.5. x2=-5.5 (00:07:01:20)
	"I'm just looking at the err... these turning points so if it zero it is minus three" (00:08:30:14)

(vii)

Graph-wise	*"I'm just looking back at the screen to look at the coordinates" (00:09:16:00)*
	"Err... I'm just looking at the... Err... Basically the solutions are reflected around the coordinates of (a, 0)" (00:10:04:14)

(viii)

Numeric-trial	*"So that... the solutions are the same distance on the right side of (a, 0)" (00:10:16:15)*
	(wrote) P1 (0, 0, 2.5) P2 (-5.5, -3, -3) a(-1.5, 0) x1 = 2.5. x2=-5.5 (00:11:43:00)
Numeric-trial	*"I'll probably write it in algebraic way..." (00:14:08:24)*

(ix)

Graphic-numeric	*"No... I'm happy with what I have written there..." (00:14:25:19)*

P15's chosen strategies in the Root task in Static

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Graphic-numeric	*00:00:09:07 "You've got your 2 point not and the one at 2 and a half"*	
Graph-wise	*00:00:32:17 "Ah ok! I see what the rotation means now... So you gonna get one there... one there... So your point not moved to three"*	
Numeric-algebraic	*00:01:16:05 "When a is 2 I'm expecting then always we have..."*	*(a,0) = (1.5,00) goes to -3 2.5 goes to-2a -5.5 -2.7; -1, 0.73; a=2 3.3; 5.0; 6.7*
Graph-wise	*00:02:14:24 "I'm just thinking kind of... how they move... and they are moving in relation to that point there... so it has something to do with the distance from... the point to a" "So kind of a way of thinking about that but it is not that ((erased -2a)" "It's the same, obviously"*	
Numeric-trial	*00:03:45:11 "The distance from plus to here is 4.7"*	
Numeric-algebraic	*00:03:52:03 "You kind of have the distance between x and a"*	
Graphic-algebraic	*00:04:09:03*	*(wrote) distance from x to a a-x is same*
Numeric-trial	*00:04:25:20 "That one is presumably the number of significant figures it can show"*	*(wrote) 2-0.73=1.27*
Numeric-trial	*00:04:52:09*	*(wrote) 2+1.27=3.3 "OK"*
Numeric-algebraic	*00:05:16:21*	*(wrote) 2-x is a-x*
Algebraic-manipulation	*00:05:40:14*	*(wrote) a+(a-x) (wrote) y=2a-x ((wrote)*
Algebraic-manipulation	*00:06:05:23 "Let's see if it works"*	*Checks flips back to previous page)) a+a - x is 2a - x(wrote) a= -2; x=0; y=-4*
Algebraic-manipulation	*00:06:21:04 "Hmmm! Let's go to the next one"*	*00:06:20:22 (wrote) a =6; x= 0 ; g = 12 x=5 g = 7*
	00:06:57:08 "7 and 12. WHOA!!! I can do maths!" ((Laughs))	

APPENDIX H OTHER APPENDICES

ASSIGNMENT OF TASK AND INSTANTIATION

Code	REP	Task	Code	REP	Task
P1	S	R	P2	S	C
	S	T		S	R
	S	C		S	T
P3	D	C	P4	D	T
	D	R		D	R
	D	T		D	C
P5	I	R	P6	I	T
	I	T		I	R
	I	C		I	C
P11	S	C	P16	S	T
	D	T		D	R
	I	R		I	C
P10	S	T	P12	S	R
	I	C		I	C
	D	R		D	T
P13	D	C	P15	D	C
	S	T		S	R
	I	R		I	T
P8	D	R	P18	D	T
	I	C		I	R
	S	T		S	C
P7	I	C	P14	I	R
	S	R		S	C
	D	T		D	T
P17	I	T	P9	I	T
	D	R		D	C
	S	C		S	R

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Institute of Educational Technology

PERSONAL INFORMATION

Date :

Name:

Age: Sex: Participant Code:

Filename codes:

Instantiation type: S – Static; D – Dynamic; I – Interactive
Task type: R – root variation; C – chordal midpoint; T – tangential region
Participant type: Student; Expert; Pilot

Type of study	*Study name*	*Stimuli name*
Trial	Trial study	Static fixed point
Pilot	Pilot study	Static root

Filenames:

TASK	*EYE-tracking AVI*	*Worksheet Tablet AVI*	*Video AVI*	*Worksheet Journal*
Trial				
Static root				

Tasks given (in order)

TASK	*Type*	*Date*	*Time Started*	*Time Ended*
Trial task				
Task 1				
Task 2				
Task 3				

CONSENT FORMS AND RECEIPTS FOR COMPENSATION

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Fax (01908) 654173

Institute of Educational Technology

I have examined the information describing the study in which you would like me to participate and herein express my consent to do so with the following provisions:

1. I can withdraw from the study at anytime for any reason;
2. That video and audio recordings will only be used for the project; and
3. Any personal data will be kept confidential and other form of data will be anonymised.

Participant's Name:

Signature:

Date:

This is to acknowledge that I have received the amount of £xx.xx. after participating for xx hours.

Participant's Name:

Signature:

Date:

DETAILED TRANSCRIPTS

P15's strategies for Static-Root task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Graphic-numeric	*00:00:09:07 "You've got your 2 point not and the one at 2 and a half"*	*She looked at the numerical solutions and looked at the graphical solution.*
Graph-wise	*00:00:32:17 "Ah ok! I see what the rotation means now... So you gonna get one there... one there... So your point not moved to three"*	
Numeric-algebraic	*00:01:16:05 "When a is 2 I'm expecting then always we have..."*	*(a,0) = (1.5,00)* *goes to -3 2.5 goes to-2a -5.5* *-2.7; -1, 0.73; a=2* *3.3; 5.0; 6.7*
Graph-wise	*00:02:14:24 "I'm just thinking kind of... how they move... and they are moving in relation to that point there... so it has something to do with the distance from... the point to a" "So kind of a way of thinking about that but it is not that ((erased -2a)" "It's the same, obviously"*	
Numeric-trial	*00:03:45:11 "The distance from plus to here is 4.7"*	
Numeric-algebraic	*00:03:52:03 "You kind of have the distance between x and a"*	
Graphic-algebraic	*00:04:09:03*	*(wrote) distance from x to a a-x is same*
Numeric-trial	*00:04:25:20 "That one is presumably the number of significant figures it can show"*	*(wrote) 2-0.73=1.27*
Numeric-trial	*00:04:52:09*	*(wrote) 2+1.27=3.3 "OK"*
Numeric-algebraic	*00:05:16:21*	*(wrote) 2-x is a-x*
Algebraic-manipulation	*00:05:40:14*	*(wrote) a+(a-x) (wrote) y=2a-x ((wrote)*
Algebraic-manipulation	*00:06:05:23 "Let's see if it works"*	*Checks flips back to previous page)) a+a - x is 2a - x(wrote) a= -2; x=0; y=-4*
Algebraic-manipulation	*00:06:21:04 "Hmmm! Let's go to the next one"*	*00:06:20:22 (wrote) a =6; x= 0 ; g = 12 x=5 g = 7*
	00:06:57:08 "7 and 12. WHOA!!! I can do maths!" ((Laughs))	

P4's strategies for Dynamic-Root task

Strateg(ies)y	*Utterance(s)*	*Description of gazes/sketches/actions*
Gesture-drawing	*00:00:55:22 "It is going to go... If it rotates a hundred eighty degrees it is moving this way... moving this way... I think it is not going to change so much"*	*P4 did a kind of circular movement with her hands.*
Graph-wise; Visual	*00:01:15:04 "It is going to be the same thing. I'll write that... I guess that is something interesting to know... same thing*	*After describing the behaviour of the graph she drew some graphs about her inference*
Graph-wise; Gaze-drawing	*00:01:33:12 "Let's say if I move this to one zero and I rotated 180 degrees it just got shifted one point to my left. So I'm guessing if I go two zero..."*	*She is describing the behaviour of the graph using her eyes as shown by her eye-movement.*
Gaze-drawing	*00:02:05:08 "It shall be shifted two points... Yeah!"*	*She is predicting the new point on the graph that is not there being reflected by the movement of her eyes.*
Visual	*00:02:14:21 "So If I go one zero, it gets shifted to...this went to four zero ok"*	*She drew some graphs about her inferences*
Graph-wise	*00:02:44:23 "that's right I guess the same thing will happen if I do the other way around... Good"*	*She stated that the same effect on the graph's behaviour as her previous inference.*
Graph-wise	*00:03:04:13 "Ok So let's see if I had this ... it moves up... if it is zero zero... it doesn't stay the same... that is because"*	*P4 described that the graph is moved upwards.*
Graphic-algebraic	*00:04:00:05 "X^3 + 2 and if I rotate around (0, 0)... it goes down this way and that becomes x^3 -2... That's interesting..."*	*She related the equation with the graph's behaviour*
Gesture-drawing	*00:05:18:09 "If I shifted two zero just like the rest of them over there... yeah"*	*She changed the point of rotation to (1, 0). P4 pointed to the screen then she moved her hand as if drawing a cross using her hand.*
Point-wise	*00:05:25:01 "It is still two zero but it got shifted two point that way"*	*(She is relating the rotation in terms of the turning point which is a discreet point of a cubic function)*
Graph-wise	*00:05:57:10 "That's something... if it is minus one... It shifted to your left... let's see..."*	*She described the behaviour of the graph*
Graphic-algebraic	*00:07:15:04 "Ooh! I can't see what's happening here" She changed the zoom scales... and looked at the graph. I don't know what I'm doing"*	*She expanded (X+2) ^3 then she entered x^3+ 6x^2 +8.*
Gesture-drawing	*00:08:57:12 "I reckon if you rotate it will be the same although I can't test that Err if I shift it two points that way I only have one point."*	*Reread the problem. (moving his hand towards the left and making a point)*
Textual	*00:09:45:13 "I don't know what you call that but I think it is inflection."*	*She generalised her conjecture speaking out loud what she is writing.*

P6's strategies for Interactive-Root task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Algebraic-chunking	*00:01:36:02 "Err... Looking at the old function and now looking at the actual equation I mean the new function it's the same aside from the fact that the sign has changed from the actual value..."*	*He looked at the equations and compared the signs of each term.*
Numeric-trial	*00:02:14:12 "It's just... yeah it's just reversed and the signs have changed... so it's flip and the signs have changed to its value to the opposite..."*	*He looked at the numbers and described the numbers.*
Graphic-numeric	*00:02:23:22 "I'm gonna move the rotation to something... I'll move it to minus two to keep it to integer... I can see what it has done... everything has shifted down to minus two... now ignoring the actual equation... "*	*He looked at the number and related it with the graph.*
Graph-wise	*00:03:07:07 "Ok... I'm gonna put it back in zero... you've added two... so there everything has two added onto it... but obviously the coordinates of the is two.. but having said that... the difference between ... the first two points is two... yeah the difference is two"*	*He described the coordinates of the graph*
Numeric-trial	*00:03:45:17 "Let's move it over to the outside... yeah... now everything has been shifted six... so six has been added to all of there... you are always adding a value...and that value is the same for each solution."*	*He did some computation wit the numbers. He read the question.*
Graphic-numeric	*00:04:33:12 "There I've added eight onto everything... so you're just adding a constant to all of those... let us go back to zero... so I added two onto everything... I can see why that is... because this point here is that point here and the difference of that is two..."*	*He was relating the graph with the numbers.*
Numeric-trial	*00:05:37:13 "I'll try with just a random... again I've added a constant to all of these... in this case I've added eight to everything..."*	*He worked out the numbers*
Graphic-numeric	*00:06:07:09 ""When it is zero... everything shifts two... when it is two everything shifts six..."*	*He was looking at the graphs behaviour and relating it to the numbers.*
Numeric-trial; textual	*00:06:16:07 "... shift equals two... hmmm... interesting"*	*He wrote on the Tablet PC in a tabular like form.*
Numeric-trial	*00:07:02:09 "So four... I've just done this... I wonder if... I've done this on*	*He evaluated the numbers he listed down.*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	purpose... so when I shifted it to position three everything shifted by eight... by five everything shifted by ten... I'm wondering whether by four it added on nine to everything... Oh no... it is twelve innit..."	
Graphic-numeric; textual	*00:07:56:19 "but the question is... what can you infer... so I'll get more information on that.... yeah if I'm shifting about the middle point... then I'll end up with the same... so if I shift it plus one... the difference now between all those Roots is plus two... plus three and now it's plus six..."*	
Graphic-numeric	*00:09:55:19 "...I shifted it one it went of two... three... six... by four to eight... obviously plus two would have been four... it means that all the Roots would adjust... by.... by double that... so if I adjust the point of rotation... one unit from the middle Root.... everything shifted by plus two... "*	*He was describing the graphs behaviour without changing the zoom but rather he was looking at the numbers*
Graphic-numeric	*00:11:43:15 "Oh... alright... that should be plus twelve... actually that is only four so plus eight... if I put that... that's better... it is a bit clearer now... from the middle Root... I moved the rotation point one unit down... that should mean that the Roots will be two less than that... than they are... minus five should be ten less here... yeah...down four..."*	*He kept on relating the graphs with the numerical computations he was doing.*
Graphic-numeric	*00:13:09:18 "I've got that... but changing the size of the graph maybe it has a big effect... so now... I might be right... that should shift two onto everything... actually it shifts four... it shifts four because this shape of the graph is changed... Non I'm confused... Err..."*	*Relating the change in the numbers because of the shape of the graph*
Graphic-numeric; gesture-drawing	*00:14:23:08 "It's kind of confusing... I'm just gonna have a play around... the Roots are two one..."*	*His eye movement still looking at the numbers and the graphs whilst whispering some computations... He counted using his hands.*
Graph-wise; mouse-drawing	*00:15:34:06 "I think I might got have any lead now... It doesn't matter... the shape of the graph.... I got minus three... zero and one... as my inflection... err... as my Roots yeah... "*	*He concentrated on describing the graph in terms of the Roots movement.*
Graphic-numeric	*00:16:36:23 "I'll just make the graph more manageable... just make it that... that's minus one that's one is the middles value and that's zero... and I'll have that so what I've done is that I added four*	*He figured that the numbers are not in order and related it back to the graph.*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	onto that one... three onto that one and five... one two three... it is confusing me a bit... because the order of this is not the order of these..."	
Gesture-drawing	*00:17:40:22 "Err... it's not that... so it is symmetrical about the rotation point..."*	*His hands were moving working out the symmetry.*
Mouse-drawing	*00:18:12:09 "... and I have just taken four off that first one... which is one two three four..."*	*He used the mouse pointing at the number and counting using the mouse pointer.*
Graphic-numeric; mouse-drawing	*00:18:25:24 "and then taken five on that one... it is the one... one two three four... oh sorry... that's zero zero... one two three..."*	*He is counting using the mouse. he related the graph to the numbers*
Graph-wise; mouse-drawing	*00:18:49:03 "one two three four... I am getting a bit confused... that's minus one and that's three... that's zero and that is four... it seems obvious... it is just rotating it... obviously everything just shift the same amount about that point..."*	*He kept on using the mouse to count the numbers. He described the Roots in terms of distance from the rotation point.*
Graph-wise; gesture-drawing	*00:20:30:16 "Yeah you (made a sound... it is just that..."*	*Whilst making the sound he moved his hand as if rotating his hand.*
Graphic-numeric	*00:20:36:06 "it's just... if that one there... the rotation point two... it is just two and add the one..."*	*He was using the mouse to point on the numbers. Counting using the mouse. He was relating the rotation with the numbers and doing some operations.*
Graphic-numeric	*00:21:08:13 "It is three... no it's two minus minus one... so just add that on... to get the five... so the graph is symmetrical... that flip over and add on four to seven... so it's that one..."*	*Describing the graph and doing some calculations.*
Graph-wise; symbolic	*00:22:11:00 "a not plus... so these are the Roots... the Roots are x y z... so it is just the difference between a not and each of the Root... is the amount you add on or take away to get the new position..."*	*He tried to record symbols but decided to describe the graph.*

P18's strategies for Static-Chord task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Graph-wise	*00:00:14:12 "I'm just looking at the lines... the boundaries... and I'm looking at that one there and seeing what ..."*	*He described the graph using the colours*
Mental-drawing	*00:01:35:02 "I'm just looking at between these two... I'm trying to visualise in my*	*It hard to tell from the gazes that there are lines being drawn.*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	head where all the midpoints could be..."	
Gaze-drawing	*00:01:49:03 "So... Definitely in this area... underneath the... surrounded by the yellow line and the cubic... definitely there is gonna be midpoints in there..."*	*Lines are being drawn in the area using eyes*
Pen-drawing	*00:02:22:04 "I'm just gonna draw on here... I just want to get an idea of where the midpoint would be... I'll just draw a bit bigger to see more detail..."*	*Lines were drawn on the Tablet PC.*
Gaze-drawing	*00:04:24:04 "I'll have a look at this next one... I can see on this one that again... from each of the midpoint to the maxima... the verti... the horizontal line I mean...come up there... Tangent underneath bounded by the cubic itself... there is midpoint in there... but also..."*	*Lines being drawn with the gazes.*
Point-wise; gaze-drawing	*00:05:20:01 "you can have midpoints higher than than... because I think you can get a Chord from the top of the maximum right up this line here... the cubic"*	*He is relating it the maxima describing where the midpoints lie. His gazes drawing some lines.*
Gaze-drawing	*00:05:38:19 "obviously the same... down with the minimum... between the minimum and the cubic as well... I'll have a quick flick around to have a look at the others...right... I'm just having a look at the functions and trying to work out in my head... where the area of the midpoints..."*	*Lines are being drawn in the different slides.*
Gaze-drawing	*00:07:02:04 "I reckon this one as well... I'll have a look at the next set..."*	*Lines a re being drawn.*
Gaze-drawing	*00:07:39:11*	*Lines are being drawn.*
Graphic-algebraic; gaze-drawing	*00:08:08:22 "I just go and look at this vertical... alright I get the general idea... where the midpoint look... like... where they can be... I'm just thinking if I can make an equation... you have the equations up there..."*	*Lines are being drawn. He wants to relate he graph with the equation of the cubic.*
Pen-drawing; visual	*00:08:49:03 "I'm just gonna draw... one of the... just draw a cubic on here and shade in an area where I think you can get midpoints... probably draw some*	*Draw a cubic shade some region and then*

Strateg(ies)y	Utterance/s	Description of gazes/sketches/actions
	lines with numbers on... just see if I can..."	
Graph-wise	*00:10:16:07 "two lines bound to this shaded area... I think it would be parallel... if this is symmetrical function then it would be parallel... the lines... the minimum and the maximum would be stretch in that way..."*	*He described the graph*
Point-wise; visual	*00:12:34:24 "I am drawing the quadratic... the cubic... I'm labelling the maximum and minimum p and q... just you can see where they are... I will draw a line from each one of hem... that goes of to he cubic ends..."*	*He drew and related the boundary to the maximum and minimum.*

P4's strategies for Dynamic-Chord task

Strateg(ies)y	Utterance/s	Description of gazes/sketches/actions
Gaze-drawing	*00:02:37:24 "I'm not certain what this means..."*	*The participant looked at the graph*
Mouse-drawing	*00:03:40:17 "That's a Tangent line... ok... equation of the vertical line..."*	*P4 pointed at minus zero then at point six using the mouse. Then she kept moving the mouse pointer*
Graph-wise	*00:05:04:05 Yeah... I know that thing... but... the midpoint goes up... it goes up..."*	*She continued animating the graph. She concentrated looking at the graph*
Mouse-drawing	*00:05:19:17 "Let me just try something easy... (Inaudible)... I can stop that... Zero zero that's the easies... It's the region that moves with this"*	*She inputted different numbers. Then, she animated the graph. She used the mouse and pointed at (0, 0). She moved the mouse pointer downwards and upwards*
Graphic-numeric	*00:07:50:19 "If I increase the speed maybe I'll know... I don't know it looks... A bit likes... asymptotic to it... (Inaudible) ...whether it's gonna be asymptotic to it... it is not moving out... right...*	*She animated the graph again. The graph moved downwards. She looked at the graph and the numbers back and forth as she watched the animation*
Visual	*00:08:32:20 Let us assume something... If I have line from here to here and moving this way it goes asymptotic... asymptotic... If I have it this way zero zero to here it goes this way it is also asymptotic..."*	*Left part of the worksheet) A cubic graph with an arrow pointing towards the left. Wrote "asymptotic"* *(Right part) A cubic with an arrow drawn going to the right. Wrote "asymptotic"*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Mouse-drawing	*00:08:55:10 "let me try something... zero zero and... it move just this way... let's move down"*	*She looked back at the interface. She pointed at (0, 0) and moved the mouse pointer downward*
Graph-wise; Textual	*00:09:10:05 "So all it does... So all it does... comes down comes down... until it becomes Tangent line zero and it changes direction... yeah..."*	*She animated the graph and watched it. She looked at the Tablet PC. She wrote on the Tablet PC. Drew an arrow pointing left and wrote "Tangent"*
Graphic-numeric trial	*00:09:37:00"Alright... Ok... and then if I used one to minus one... (Inaudible) (So it just moves along) Ah! Ok...using minus one to one... It goes down that way... So it crosses the line... But... it cuts one zero... So that's down right...and it still keeps going...How long can this go to... I'm still trying to figure out what's going on...That keeps going... that just never...When does the midpoint end..."*	*She animated the graph in different directions. She looked at the graph and numbers back and forth as she watched the graph. She changed the zoom scale*
Graphic-numeric	*00:13:14:00 "It gets to... who care... minus six let's go back to that... This is minus...Tangent is here... it goes up... so obviously... it's doing that... I understand that one... zero... it would draw a line there... It is going to fast... It reaches the tangency... It goes down... it goes down... it reaches zero and then... let's stop there... The midpoint is minus zero point nine minus ten point five..."*	*P4 gasped. She animated the graph in every direction. She changed the speed of animation. She looked at the graph and numbers whilst watching the animation of the graph*
Mouse-drawing	*00:14:13:22 "It will never ever comes cross... Something... it never comes across..."*	*She then stopped the animation then pointed the mouse at the region inside the curve and the graphical line. She moved the mouse pointer*
Mouse-drawing	*00:15:01:03 "One point five and..."*	*She used the mouse and moved the mouse pointer from 1.5 going left and right*
Gaze-drawing	*00:15:49:00 "What will happen is that... it goes through..."*	*She looked back at the screen then animated the graph.*
Graph-wise	*00:16:03:17"It goes to zero zero... It then goes through... It becomes a Tangent to it... It shifted... it shifts... and then it becomes asymptotic to something which I can't figure... but it is... Is it the point it self one zero... It could be the point itself one zero... because it never ever comes across"*	*She continued animating and watching the graph. She looked at the graph and occasionally looked at the numbers.*
Pen-drawing	*00:16:48:05*	*(Right-most part of the Tablet PC) Drawing line from (0, 0) going upwards.* *(Bottom part of the Tablet PC) Drew an XY plane. Mouse pointer*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
		making some lines.
Gaze-drawing	*00:17:11:01*	*P4 glanced back on the computer screen.*

P12's strategies for Interactive-Chord task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Mouse-drawing	*00:00:00:00 "So probably... I'm trying to visualise the area... I wonder if you can get those points... probably you can get all of them... Err"*	*She moved the mouse pointers as if line was being drawn along the cubic curve.*
Visual	*00:01:11:12 "... So I've got this... and I reckon you can get an area all the way... something like that..."*	*She drew a cubic curve on the Tablet PC and shaded some regions.*
Mouse-drawing	*00:02:40:17 "the vertex of the graph and... I suppose it go on to infinity maybe although would I even get up there... I suppose I'm imagining that you can get midpoints in here but if it goes up here would I actually..."*	*P12 was moving the mouse like shading some regions and drawing some lines.*
Pen-drawing	*00:03:03:14 "you get one here... I suppose in this graph you wouldn't... Ok I'm gonna do a bit more drawing... so if I..."*	*She drew some dotted lines and erased it afterwards.*
Pen-drawing	*00:04:50:01 "Something very vaguely like that... but if I fix that and go down here so they get..."*	*She drew some dotted lines of where the midpoints are traced from the graph..."*
Pen-drawing	*00:05:50:19*	*She added more dotted lines on her drawings.*
Visual	*00:05:58:03 "So this are like my boundaries for when I am moving different ends of the midpoint and I reckon I can get all of that area...Err... Argh... "*	*She shaded some region of the graph she drew.*
Graph-wise; pen-drawing	*00:06:27:19 "a bit messed up... Ok... Hmm..."*	*She erased some bits of the region she had shaded.*
Visual	*00:09:14:01 "They are a bit more...I wonder of you can get any midpoint in this bit... Hmmm..."*	*She drew another curve and shaded some regions.*
Graph-wise	*00:11:07:15 "The top boundary is kind of determined by... I wonder if I can get... Oh... it is probably more complicated... I bet you can get all of the points...I*	*She concentrated on describing the behaviour of the graph*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	suppose it is difficult... "	
Pen-drawing; visual	*00:15:00:18*	*She read the problem out loud. Then he drew another curve. She was drawing what she could trace with the use of the Interactive software.*
Visual	*00:15:53:07 "So there is a boundary between the midpoints in the curve... so it is between there and between there..."*	*She continued sketching on the graph tracing the midpoints movement.*
Visual	*00:16:43:02 "And then similarly... I suppose it is around here"*	*She added more drawings on the curve.*
Pen-drawing; visual	*00:18:01:01*	*She drew another curve and again started shading some region where it is being traced by the midpoint using the interactivity.*
Visual	*00:19:34:07*	
Pen-drawing; visual	*00:19:44:18 "Hmm... Ok... Err..."*	*She went on shading her drawings. The pen stylus pointer was moving making some lines being drawn.*
Graph-wise	*00:20:13:11 "I think I'm kind of done...*	*She described the behaviour of the graph by sketching.*

P1's strategies for Static-Tangent task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Mental-drawing	*00:06:20:02 "Ah ok... so by moving that point from there to where it is in b... the pink Tangent gets much steeper... there green one more... as soon as you cross this... line the green line would no longer exist... and same with the pink one... so that determine he intersection with the graph..."*	*There were no lines being drawn.*
Mental-drawing	*00:07:36:05 "So when it is on that side of the graph... you get one... on that side you get three... as soon as you cross this line... the pink one no longer exist and the blue line still there..."*	
Graph-wise	*00:11:29:06 "So that point moved only in the x direction..."*	*She described the behaviour of the graph. The movement of the point.*
Point-wise; gaze-drawing	*00:11:44:15 "As you moved across this maxima... the pink one... turns... as soon as you move to the right of it you get one going in the opposite direction... hmmm... that blue line... as you go across that one would disappear... "*	*Lines being drawn.*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Point-wise; gaze-drawing	*00:12:20:12 "The blue one also sort of the same... that just changes gradient... until you went across the top... that one would disappear... and the same... wit the green one I think... it is gonna move further up there... so it becomes inflected so it no longer... becomes a Tangent..."*	
Point-wise	*00:13:28:04 "I'm just trying to work out... the turning point reflected the regions... or something like that..."*	*The gaze constantly going to and from the minima.*
Point-wise; gaze-drawing	*00:14:12:18 ""If we move from there... the pink one cannot exist as soon as you go through the actual lines of the graph... I'll look at a different one... just one moves along the x-axis as well... so "* *Gaze and relate it tot the minima...*	*Gazes are being drawn from the minima.*
Gaze-drawing	*00:16:16:04 "it moved to the right... that one would exist until you go trough... that line... and then it will just become Tangent to that curve instead... within that part of the graph... you still have one... and then when it gets outside you get three again..."*	*Lines being drawn...*
Graph-wise	*00:17:02:18 "... I'm just thinking about whether... whether the x coordinate of the point... is inside or outside of the line of the graph... when it's... obviously inside... in got one... outside it got three..."*	*Described the behaviour of the graph.*
Visual	*00:18:58:06*	*She drew a cubic with lines from the maxima and minima. She redrew a bigger graph again lines from the maxima and minima*
Point-wise	*00:20:20:17 "I'm trying to draw... what I think might be a boundary... but I need to check if that is correct... so at the moment... if trying to... that it is bounded in the y direction y the maxima and minima... and in the x direction by the outside Roots of the equation... hmmm it doesn't work... (laughs)"*	
Point-wise	*00:20:57:04 "No it doesn't work... yeah it does work for that... but it is not for the... Roots... I decided that it has something to do with the line of the graphs... and it has something to do with the turning points..."*	
Gaze-	*00:21:55:19 "...Hmmm... that doesn't*	*Gaze drawing lines*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
drawing	*work... sorry... (laughs)..."*	
Mouse-drawing	*00:23:12:07 ""outside there you will have one... two three... "*	*Moving the mouse counting the number of Tangents.*
Gaze-drawing	*00:23:22:00 "so that...one... two... three... on the graph... err..."*	*Lines being drawn.*
Graph-wise	*00:24:14:12 "It looks like... that outside the graph you have three... but I need to... find some actual rule..."*	*Described where the boundary she thought.*
Gaze-drawing	*00:24:37:22*	*Lines are being drawn. The researcher reminded the participant to keep on talking.*
Gaze-drawing	*00:25:53:24 "... I'm trying to find... it looks like that point... I wonder if tends to infinity... no because it would move to... to the right... I can't think of ways of defining it... (laughs)"*	*Lines were being drawn.*
Visual	*00:27:00:04 "Inside that... where that changes... that's as soon as you cross this line...the same up there... it will change because of the line..."*	*Drew on the tablet.*

P4's strategies for Dynamic-Tangent task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Gesture-drawing	*00:00:53:19 "The point of inflection itself... which is..."*	*P4 pointed at the inflection and moved his hand toward her.*
Graph-wise; Visual	*00:01:52:08 "at minus one... it is the same thing except in the opposite direction"*	*She drew a graph with a Tangent She described where the position of the Tangent line.*
Mouse-drawing	*00:02:10:13 "Ok if I choose a point let's say zero one... sorry... one zero"*	*She was moving the mouse point before making her decision of what to enter.*
Textual	*00:02:45:13 "Zero one zero we have to point of tangency"*	*She wrote (1, 0) and 2 points on the Tablet PC.*
Graph-wise	*00:02:58:17 "zero one... alright so I'm guessing that for minus one it's the same thing... so that is going down...Err... I'm going to try something down here"*	*P4 entered zero one. She described the position of the Tangent.*
Graph-wise; Visual	*00:03:42:24 "three minus five... Oh! Ok minus three minus five... which is here somewhere... which is somewhere near the*	*She -3 and -5. She drew a graph and plotted a point and a Tangent line and wrote 3 points and the signs of*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	inflection we have three points"	*the coordinates.*
Graph-wise	*00:04:25:16 "When I did it this way... we had an inflection over here... I can't remember."*	*She described where the location of the Tangents appeared.*
Mouse-drawing	*00:06:00:03 "Ok. Let us try something over here... three minus five..."*	*She moved the mouse pointer projecting where the points are going to appear.*
Gaze-drawing; Textual	*00:06:12:09 "Three minus five there is one Tangent... Hmmm! Ok!"*	*After animating, her eye movement is drawing some lines being drawn towards the cubic.*
Gesture-drawing	*00:06:40:17 "So on that side... I'm guessing we gonna have three here..."*	*She moved her hands as upwards and downwards as if doing some lines.*
Gesture-drawing	*00:07:42:24 "... I got two points... when I did it over there at the end... one on that side... so..."*	*Her hands were moving left and right up and down.*
Graph-wise; Visual	*00:08:22:10 "If it is in this region and this region we have three Tangents... if we have it here we have two Tangents...hmm..."*	*She drew a graph and shaded some regions. She described her drawing.*
Gaze-drawing	*00:09:03:18 "If I have..."*	*P4 gazes where some straight lines were being drawn exactly along the y-axis.*
Visual	*00:09:12:10 "...minus zero point one... minus five... Ha! These are not two Tangents... that is one Tangent... so"*	*She erased what she previously wrote.*
Textual	*00:10:21:12 "I don't know what you call that... Err... it's... err... if the line passes through..."*	*She wrote on the Tablet PC speaking aloud what she was writing.*
Gaze-drawing	*00:11:15:16 "That's horizontal line thingy..."*	*Her gazes depicted some lines being drawn.*
Textual	*00:11:18:24*	*She continued writing on the Tablet PC.*
Gesture-drawing	*00:12:29:23 "if the point...minus one lies one the right..."*	*Her hand moving making a line.*
Gaze-drawing	*00:12:37:00*	*The movement of the eyes drawing some lines.*
Gesture-drawing	*00:12:41:17 "lies one the right... It doesn't make sense... lies on the..."*	*Speaking aloud what she is writing. Her hands movement toward the right.*
Graph-wise; Textual	*00:12:58:04 "right of the upper portion... or on the left of the lower portion there are three Tangent"*	
Textual	*00:13:49:18*	*She continued writing some text on the Tablet PC about her inferences.*

P15's strategies for Interactive-Tangent task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Pen-drawing	*00:00:31:04 "So imagine we got a curve... so this..."*	*The stylus movement drawing a line.*
Graphic-numeric trial	*00:00:38:06 ""see where it goes back so point three three... there's a point in there..."*	*She is relating the graph with the coordinates of P.*
Visual	*00:01:15:14 "Somewhere up here we have three Tangents..."*	*She drew a graph and... and a Tangent line*
Visual	*00:01:34:04 "We got one"*	*She continued sketching on her drawing*
Visual	*00:01:55:09 "And that would be three"*	*She added another Tangent line on her drawing.*
Mouse-drawing	*00:03:03:13 "The area outside which is there... What's there though... what is the point"*	*She moved her mouse pointer like lines Tangent to the curve were being drawn.*
Graphic-numeric trial	*00:03:20:10 "I can't work out what the Tangent is when it switches from three to one... "*	*She is trying to relate the graph with the coordinate of P as shown in her gazes.*
Mouse-drawing; Visual	*00:04:06:00 "I don't know how this works though... it is just this Tangent there... there is a point... when you can't have any Tangent to this... and in here you have... no Tangent and in that middle bit you have Tangents to inner... up here"*	*She moved the mouse pointer like lines being drawn. She added more notes on her drawing.*

P15's strategies for Dynamic-Chord task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
	00:00:10:05 "I'm just trying to think of really simple... quadratic... cubic"	*(typed) x^3 + 4x^2 + 5x + 4"*
	00:00:36:10 "I'm thinking of points to put in... like O, four"	*(Typed) 0.0; 4.0; -2.0; 2.0*
Gaze-drawing	*00:00:54:01 "what can you do with the straight line"*	*P15's eye movement were drawing some lines.*
Gaze-drawing	*00:01:36:19*	*P15's eye movement were drawing some lines.*
Graph-wise; Mouse-drawing	*00:01:41:10 "I'm just thinking as that lines stretches up there and as that line stretches down there your midpoint..."*	*She moved the mouse pointer drawing some lines on the screen.*
Gaze-drawing	*00:02:00:00 "The midpoints are always going to be within the two lines of the curve"*	*Her eye movement made a line from one point of the curve to the other.*
Pen-	*00:02:13:13 "So when the shape is like*	*She used the pen to draw a cubic.*

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
drawing	*that... it is not going out there... so it is going to be in the area... can you have..."*	*Then the stylus pointer was making some lines being drawn before she actually draw the line.*
Gaze-drawing	*00:02:47:19 "Can you get midpoint right on there... yeah you probably can."*	*P15's gazes lines being drawing.*
Gaze-drawing	*00:03:22:07 "You can't get midpoints all the way down to..."*	*Gazes drawing some lines outside the shaded area she drew.*
Gaze-drawing	*00:06:29:23 "so what happens if we take..."*	*Again lines were being drawn by her eye gazes.*
Pen-drawing	*00:06:34:14 "So when it goes up there... you can have... when it comes down you feel this space..."*	*Pen making some lines before exactly making the actual drawing.*
Gaze-drawing	*00:07:06:22*	*Some lines being drawing with her eye gazes.*
Pen-drawing	*00:07:09:15 "yeah ok..."*	*Small sketch was made and was erased immediately.*
Gaze-drawing	*00:07:26:04*	
Gesture-drawing	*00:07:33:15 "I think if you put in a vertical line... that's gonna go say two..."*	*Her hands moving as if drawing some lines.*
Gaze-drawing	*00:08:20:00 "that one goes up there... and your Chords gonna be getting closer to your curve... and the it will go outside of the curve"*	*Her eye movement was going toward the spaces where she is saying before the animation goes there.*
Pen-drawing	*00:08:41:14 "So it is not quite the area bounded by the curve because you can't get a Chord out here"*	*The stylus defining the region where Chord cannot appear before she shaded it.*
Gaze-drawing	*00:08:52:23 "What am I Looking for... I'm looking for the boundary..."*	*Still doing some lines.*
Graph-wise; Gaze-drawing	*00:09:55:05 "... The cubic goes on forever... but if you kind of describe it like err..."*	*Eye movement still drawing some lines being drawn.*
Pen-drawing	*00:10:20:22 "If you put it in a box and you have your cubic... then it will be everything... it is not just a rectangle it is a kind of..."*	*Pen drawing ...*
Gaze-drawing	*00:11:02:17 "X cube"*	*Line being drawn.*
Pen-drawing	*00:11:05:18 "I'm trying to describe that..."*	*Pen moving like drawing lines*
Gaze-drawing	*00:15:24:22 "But it have a spare yellow line"*	*Eye gaze drawing a line on the curve.*

P3's strategies for Dynamic-Chord task

Strateg(ies)y	*Utterance/s*	*Description of gazes/sketches/actions*
Graph-wise; gesture-drawing	*00:02:00:10 "I was thinking that the midpoint was... always on the vertical... it looks... when it is going to the x-axis... it seems to be that the midpoint is always at the vertical... that's not right... when it comes down it is kind of of... that line is similar... but it doesn't seem to hold..."*	*He described where the midpoint is going. His hand movements were showing where the midpoints would go.*
Gaze-drawing	*00:03:43:20 "I think the bound... right the boundary of that area is always going to be between... between the... err... line"*	*Lines are being drawn between the y-axis and the curve.*
Mouse-drawing	*00:03:54:08 "... line of the cubic function and the y-axis... I think"*	*He moved the mouse pointer as if drawing some lines.*
Mouse-drawing	*00:05:40:21 "Oh... What I said before was wrong... because when it passed over the area here... the midpoint wasn't in the space where I was expecting it to be... but..."*	*The mouse movement is determining the area being described.*
Gaze-drawing	*00:05:54:06 "I think..."*	*The gaze drawing lines being drawn.*
Mouse-drawing	*00:06:03:22 ""If I put a line... if I put a line..."*	*The mouse movement like lines being drawn from one point along the cubic to another point.*
Mouse-drawing	*00:06:14:20 "... line at minus one... the green line... yeah the green line... the midpoint is always going to be in this area bounded by that part of the red line and that part of the green line... and then up here is... is bounded by that part of the green line and that part of the red line... ok that makes..."*	*The mouse drawing some lines*
Gesture-drawing	*00:06:48:12 "... that makes a bit more sense that it's... that it's within that area...rather than where the axis actually is..."*	*His hand movement defining the area.*
Gesture-drawing	*00:06:57:04 "I'm just looking at it... running it again and see what happens... I guess you can get a boundary at this point ... as it*	*Movement of the hands defining where the area could be shading the area using his hand movement.*
Gaze-drawing	*00:07:21:04*	*His eye movement drawing some lines with the area defined previously.*

Strateg(ies)y	***Utterance/s***	***Description of gazes/sketches/actions***
Textual	*00:07:34:21*	*He wrote on the Tablet PC about hi inference.*
Gesture-drawing	*00:10:50:08 "I'm not... hmmm... I can't quite tell from this if that dot is actually on... the right hand side of the green line... or not..."*	*The hand determining whether the dot is on the location he is expecting it to be.*
Graph-wise	*00:11:49:07 "can I zoom it... hmmm... that midpoint is somewhere... where I am not expecting it to be..."*	*He is describing the behaviour of the midpoint.*
Gaze-drawing	*00:12:47:14 "So I have to think again... I'm not sure where it is gonna... Does it go along a straight... no it doesn't go along a straight line at all..."*	*Lines are being drawn using his eyes.*
Gaze-drawing	*00:14:26:10 "I'm not sure... (laughs)... I really can't tell where..."*	*Lines still drawing some lines.*

LIST OF PUBLICATIONS

San Diego, J. P., & Aczel, J. C. (2007, March). *New approaches to researching the pedagogical benefit of representations and interactivity.* Paper presented at the CAL 2007, Trinity College, Dublin.

San Diego, J. P., Aczel, J. C., & Hodgson, B. (2004a, July). *The effects of technology on making conjectures: Linking multiple representations in learning iterations.* Paper presented at the International Group for the Psychology of Mathematics Education, Bergen, Norway.

San Diego, J. P., Aczel, J. C., & Hodgson, B. (2004b, February). *Linking multiple representations in exploring iteration: does change in technology change students' conjecture?* Paper presented at the BSRLM Conference, King's College London.

San Diego, J. P., Aczel, J. C., & Hodgson, B. (2004c, June). *"No! It didn't change... It's just faster!": Iteration, multiple representations, technology, and conjectures.* Paper presented at the 2004 Computers and Learning Research Group Conference (CALRG), The Open University, Milton Keynes.

San Diego, J. P., Aczel, J. C., Hodgson, B., & Scanlon, E. (2006a, August). *Learners' strategies with multiple representations.* Paper presented at the European Association for Research on Learning and Instruction Special Interest Group 2 Meeting, University of Nottingham, Nottingham.

San Diego, J. P., Aczel, J. C., Hodgson, B., & Scanlon, E. (2006b, July). *"There's more than meets the eye": analysing verbal protocols, gazes and sketches on external mathematical representations.* Paper presented at the Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education, Prague.

San Diego, J. P., Aczel, J. C., Hodgson, B., & Scanlon, E. (under revision). Digital approaches to analysing video data of learners' computer interactions. *Educational Technology Research and Development.*

It often does more harm than good to force definitions on things we don't understand. Besides, only in logic and mathematics do definitions ever capture concepts perfectly. The things we deal with in practical life are usually too complicated to be represented by neat, compact expressions. Especially when it comes to understanding minds, we still know so little that we can't be sure our ideas about psychology are even aimed in the right directions. In any case, one must not mistake defining things for knowing what they are."

-- Marvin Minsky --
from The Society of Mind, 1985

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Centre for Research in Education and Educational Technology (CREET)

San Diego, J. P.	The effect on learners' strategies of varying computer-based representations: Evidence from gazes, actions, utterances and sketches	2008	CREET

NO CD

ATTACHED